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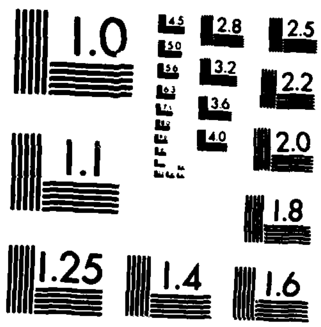

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THE BIDIMENSIONAL STEFAN PROBLEM WITH CONVECTION: THE TIME-DEPENDENT CASE

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THE BIDIMENSIONAL STEFAN PROBLEM WITH CONVECTION: THE TIME DEPENDENT CASE

J. R. Cannon[1], E. DiBenedetto[2] and G. H. Knightly[3]

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ABSTRACT

This paper considers the time dependent Stefan problem with convection in the fluid phase governed by the Stokes equation, and with adherence of the fluid on the lateral boundaries. The existence of a weak solution is obtained via the introduction of a temperature dependent penalty term in the fluid flow equation, together with the application of various compactness arguments.

AMS (MOS) Subject Classifications: 35B65, 35K60, 35K65, 35R35, 76D05

Key Words: phase-change, Navier Stokes equation, free boundary, local regularity, convection

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SIGNIFICANCE AND EXPLANATION

Consider a phenomenon (such as melting of ice) where there is a change of phase, say liquid-solid. In the liquid phase the thermal energy is transported both by diffusion and convection, and the effects of convection are reflected in the movement of the free-boundary separating the two phases.

In this paper we show that such a problem can be formulated mathematically and that it admits a solution in a weak sense.

~~We~~ also investigate some local regularity properties of the distribution of temperature and the field of velocities in the liquid phase.

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THE BIDIMENSIONAL STEFAN PROBLEM WITH CONVECTION: THE TIME-DEPENDENT CASE

J. R. Cannon[1], E. DiBenedetto[2] and G. H. Knightly[3]

1. INTRODUCTION

The aim of this paper is to extend to the nonstationary case some results obtained by the authors in [5, 6], about the Stefan problem with convection.

We briefly describe the physics of the phenomenon, referring to section 2 for a precise mathematical formulation.

Suppose that in a region Ω of R^N, $N > 2$ a liquid undergoes a change of phase at a fixed temperature. The model example we have in mind is a water-ice situation. At every time t the liquid and solid phases are determined by the knowledge of the distribution of temperature. We call $u^{(1)}$ the temperature in the liquid and $u^{(2)}$ the temperature in the solid. In general, in the liquid region there are present convective motions originated by body forces $\vec{f}$ depending on the temperature $u^{(1)}$. The dynamic state of the liquid is determined by the knowledge of the field of velocities $\vec{v}$ and the pressure p. The diffusion of heat in the liquid is affected by the velocity $\vec{v}$, and in turn $\vec{v}$ point by point is affected by the buoyancy forces $\vec{f}(u^{(1)})$.

We will describe the phenomenon of diffusion in the liquid phase by the evolution equation

$$(1.1)\qquad \frac{\partial}{\partial t}\alpha_1(u^{(1)}) - \operatorname{div} k_1(u^{(1)})\nabla_x u^{(1)} + \vec{v}\cdot\nabla_x \alpha_1(u^{(1)}) = 0$$

where $\alpha_1(\cdot)$, and $k_1(\cdot)$ represent heat capacity and conductivity respectively and are possibly nonlinear functions of the temperature $u^{(1)}$. The term $\vec{v}\cdot\nabla_x\alpha_1(u^{(1)})$ gives a description of how the velocity $\vec{v}$ affects the temperature $u^{(1)}$.

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The convection will be modeled by the system of Stokes equations

$$\frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} + \nabla_x p = \vec{f}(u^{(1)}) , \tag{1.2}$$

where ν is the kinematic viscosity, p the pressure and $\vec{f}(u^{(1)})$ the buoyancy forces.

The two equations (1.1)-(1.2) represent the classical Boussinnesq coupling of thermal diffusion and convection [15].

We assume the liquid is incompressible $(\operatorname{div} \vec{v} = 0)$. Moreover since it is viscous $(\nu > 0)$ and since we assume that the solid phase is at rest, it is reasonable to assume $\vec{v} = 0$ on the boundary of the liquid region.

In the solid phase there is only a diffusion process described by an equation like (1.1) without the term involving the velocity, since we assume zero velocity for the solid phase.

We assume the distribution of temperatures $u^{(1)}$, $u^{(2)}$ and the field of velocities are known at some initial time $t = 0$, and on the boundary $\partial\Omega$ of Ω we prescribe at every time t the heat flux g, which is a possibly nonlinear function of the temperature.

At the unknown boundary Γ separating the two phases we impose the relation $u^{(1)} = u^{(2)} = 0$ and

$$[k_1(u^{(1)})\nabla_x u^{(1)} - k_2(u^{(2)})\nabla_x u^{(2)}]\cdot N_x = LN_t$$

where $\vec{N} \equiv (N_x, N_t)$ is the unit normal to Γ directed toward the solid phase. Such a relation measures, roughly speaking, the amount of heat used in the melting process and L represents the latent heat of fusion.

The problem consists in determining at every time t, the distribution of temperatures, the field of velocities, the pressure and the configuration of the system.

Our purpose is to show that such a problem for the spatial dimension $N = 2$ admits at least a solution, in a sense to be made precise below.

We comment here on the restriction $N = 2$, and on the difficulties of extending the results of [5, 6], to the time-dependent situation.

Since the Stokes equations have to hold only in the liquid region, one has to have some topological information on the set occupied by the fluid, in order to give a meaningful interpretation to the field of velocities. For example one should know that such a set is open to view the Stokes equations at least in the sense of distributions over such a set. This information would be implied by the continuity of the temperature which in turn depends on the smoothness of $\vec{v}$. It turns out that only for $N = 2$ are we able to show that the degree of smoothness of $\vec{v}$, suffices to yield the continuity of u.

This delicate interplay between the regularity of u and the regularity of $\vec{v}$, has also prevented us from using the Navier-Stokes equations

$$\frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla_x p = \vec{f}$$

in the place of (1.2).

The limitations are quite clearly of technical nature and it is our hope to remove them in the future.

Also it should be pointed out that at this stage, uniqueness is an open question.

Sections 2 and 3 contain the classical formulation of the problem, notation and some preliminary material. The concept of weak solution is introduced in Section 4, whereas Section 5 is devoted to a listing of the assumptions and a statement of the results. In Section 6 we prove our theorem by assuming certain facts (Propositions 6.1, 6.2,), which are demonstrated in Sections 7, 8.

It is a pleasure to acknowledge conversations with Prof. B. Benjamin and W. Pritchard, on the physics of the problem.

2. FORMULATION OF THE PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. For all $t \in [0,T]$, $T > 0$ let $\Omega(t) \equiv \Omega \times \{t\}$, $\partial\Omega(t) = \partial\Omega \times \{t\}$ and $\Omega_t = \bigcup_{0<\tau<t} \Omega(\tau)$. We denote with $\partial\Omega_T$ the parabolic boundary of Ω_T i.e.

$$\partial\Omega_T \equiv \overline{\Omega(0)} \cup S_T\ , \quad S_T = \bigcup_{0<t\leq T} \partial\Omega(t)\ .$$

The set Ω_T is divided into Ω_1 and Ω_2 by the free boundary $\Gamma \equiv \Gamma_T \equiv \bigcup_{0 \le t \le T} \Gamma(t)$, where $\Gamma(t)$ is a hypersurface in $\Omega(t)$ determined implicitly by $\Phi(x,t) = 0$. The function $\Phi \in C^1(\bar{\Omega}_T)$, $\Phi < 0$ on Ω_1, $\Phi > 0$ in Ω_2 and $|\nabla_x \Phi| \ne 0$ on Γ. Here ∇_x denotes the gradient operator with respect to the space variables $x \equiv (x_1, x_2)$ only.

The set $\Gamma(0)$ divides the initial region $\Omega(0)$ into two regions $\Omega_1(0)$ and $\Omega_2(0)$. We set $S_i = \bar{\Omega}_i \cap S_T$ and denote with $\vec{N}^{(i)}$ the outward normals to S_i.

Consider the problem of determining the real valued functions Φ, $u^{(i)} : \Omega_i \to R$, a vector valued $\vec{v} : \Omega_T \to R^2$, and $p : \Omega_1 \to R$, satisfying

(2.1) $$\frac{\partial}{\partial t} \alpha_i(u^{(i)}) - \operatorname{div} k_i(u^{(i)}) \nabla_x u^{(i)} + \vec{v} \cdot \nabla_x \alpha_i(u^{(i)}) = 0 \quad \text{in} \quad \Omega_i, \quad i = 1,2$$

(2.2) $$-k_i(u^{(i)}) \nabla_x u^{(i)} \cdot \vec{N}^{(i)} = g_i(x,t,u^{(i)}), \quad (x,t) \in S_i, \quad i = 1,2$$

(2.3) $$u^{(i)}(x,0) = u_0^{(i)}(x), \quad x \in \Omega_i(0) \quad , \quad (-1)^i u_0^{(i)}(x) < 0$$

$$u_0^{(i)}\big|_{\Gamma(0)} = 0$$

(2.4) $$[k_1(u^{(1)}) \nabla_x u^{(1)} - k_2(u^{(2)}) \nabla_x u^{(2)}] \cdot \nabla_x \Phi = L \Phi_t, \quad (x,t) \in \Gamma ,$$

(2.5) $$\frac{\partial}{\partial t} \vec{v} - \nu \Delta \vec{v} + \nabla_x p = \vec{f}(u^{(1)}) \quad \text{in} \quad \Omega_1$$

(2.6) $$\vec{v} = 0 \quad \text{on} \quad S_1$$

(2.7) $$\vec{v}(x,0) = \vec{v}_0(x), \quad x \in \Omega_1(0), \quad \operatorname{div} \vec{v}_0 = 0$$
$$\vec{v}_0(x) \equiv 0, \quad x \in \Omega_2(0)$$

(2.8) $$\operatorname{div} \vec{v} = 0$$

(2.9) $$\vec{v}(x,t) = 0 \quad \text{a.e. in} \quad \Omega_2 .$$

Here the latent heat L is a known positive constant, $g_i(x,t,\xi)$, $i = 1,2$ are known functions of their arguments and $\vec{\xi}(\cdot)$ is a given vector valued function mapping $\mathbb{R}$ into $\mathbb{R}^2$. Also $\alpha_i(\cdot)$, $k_i(\cdot)$ are smooth functions defined for $(-1)^i s < 0$ and such that there exists constants γ_0, γ_1 for which

$$0 < \gamma_0 < \alpha_i'(s),\ k_i(s) < \gamma_1, \quad i = 1,2 \qquad (2.10)$$
$$\forall s \in \mathbb{R},\quad (-1)^i s < 0 .$$

In order to formulate the problem in a simple fashion we make a change of the unknowns as follows. First note that by their physical nature the heat capacities $\alpha_i(\cdot)$ are continuous, monotone increasing and coercive functions of their arguments which can be defined so that $\alpha_i(0) = 0$, $i = 1,2$. Therefore the knowledge of $\alpha_i(u^{(i)})$ determines $u^{(i)}$ uniquely. Then we define

$$u = \begin{cases} \alpha_1(u^{(1)}) & \text{in } \Omega_1 \\ \alpha_2(u^{(2)}) & \text{in } \Omega_2 \end{cases}$$

$$K(u) = \begin{cases} \int_0^u k_1(\alpha_1^{-1}(\xi))\alpha_1^{-1\prime}(\xi)d\xi\ , & u > 0 \\ \int_0^u k_2(\alpha_2^{-1}(\xi))\alpha_2^{-1\prime}(\xi)d\xi\ , & u < 0 \end{cases}$$

$$u_0(x) = \begin{cases} \alpha_1^{-1}(u_0^1(x)) & \text{in } \Omega_1(0) \\ \alpha_2^{-1}(u_0^{(2)}(x)) & \text{in } \Omega_2(0) \end{cases}$$

$$g(x,t,u) = \begin{cases} g_1(x,t,\alpha_1^{-1}(u)) & (x,t) \in S_1 \\ g_2(x,t,\alpha_2^{-1}(u)) & (x,t) \in S_2 \end{cases}$$

The equations for $u^{(i)}$, $i = 1,2$ can be formally rewritten as

$$
(2.11)\qquad \begin{cases} \frac{\partial}{\partial t} u - \Delta K(u) + \vec{v}\cdot\nabla_x u = 0 \quad \text{in } \Omega_T \\ -\nabla_x K(u)\cdot\vec{N} = g(x,t,u), \quad (x,t) \in S_T \\ u(x,0) = u_0(x), \qquad x \in \Omega(0) \\ u(x,t) = 0 \qquad (x,t) \in \Gamma \\ \{[\nabla_x K(u)]^+ - [\nabla_x K(u)]^-\}\cdot\nabla_x \phi = L\phi_t, \quad (x,t) \in \Gamma \end{cases}
$$

where $[\nabla_x K(u)]^+$ denotes the limit from Ω_1, as (x,t) approaches Γ, while $[\nabla_x K(u)]^-$ denotes the limit from Ω_2.

As for the velocities, setting $\vec{f}(u) = \vec{\underline{f}}(\alpha_1^{-1}(u))$, we can rewrite (2.5) as

$$
(2.5)' \qquad \frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} + \nabla_x p = \vec{f}(u) \quad \text{in } \Omega_1 .
$$

In order to formulate our notion of weak solution of (2.11), (2.5)', (2.6)-(2.9), we need to introduce some basic notation.

3. NOTATION AND FUNCTION SPACES

In this section we give a brief description of the function spaces employed and recall basic facts, known from the literature, to be used as we proceed.

For $q,r \geq 1$ let $L_{q,r}(\Omega_T)$ denote the Banach space of those measurable functions mapping $\Omega_T \to \mathbb{R}$ with norm defined by

$$
(3.1) \qquad \|u\|^r_{q,r,\Omega_T} = \int_0^T \|u\|^r_{q,\Omega(\tau)} d\tau ,
$$

where

$$\|u\|^q_{q,\Omega(\tau)} = \int_\Omega |u(x,\tau)|^q dx .$$

When $q = r = 2$, $L_{2,2}(\Omega_T)$ coincides with the Hilbert space $L_2(\Omega_T)$ whose inner product $(\cdot,\cdot)_{2,\Omega_T}$ generates the norm $\|\cdot\|_{2,\Omega_T} = \|\cdot\|_{2,2,\Omega_T}$. If $q = r$ we set $\|u\|_{q,q,\Omega_T} = \|u\|_{q,\Omega_T}$.

Let $W^{1,0}_p(\Omega_T)$ denote, for $p \geq 1$, the Banach space with norm

(3.2) $$\|u\|^p_{W^{1,0}_p(\Omega_T)} = \|u\|^p_{p,\Omega_T} + \|\nabla_x u\|^p_{p,\Omega_T} ,$$

where

$$\|\nabla_x u\|^p_{p,\Omega_T} = \int_{\Omega_T} |\nabla_x u|^p dx$$

and $|\cdot|$ here denotes the euclidean length of a vector in R^2.

If $p = 2$, $W^{1,0}_2(\Omega_T)$ is the Hilbert space with inner product

(3.3) $$(u,w)_{W^{1,0}_2(\Omega_T)} = (u,w)_{2,\Omega_T} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial w}{\partial x_i}\right)_{2,\Omega_T} .$$

With $W^{1,1}_2(\Omega_T)$ we denote the Hilbert space with inner product

(3.4) $$(u,w)_{W^{1,1}_2(\Omega_T)} = (u,w)_{W^{1,0}_2(\Omega_T)} + \left(\frac{\partial u}{\partial t}, \frac{\partial w}{\partial t}\right)_{2,\Omega_T}$$

Here $\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial t}$ denote generalized derivatives. These definitions are modified in the usual way if q,r,p are infinity. For $p \geq 2$ let $\overset{\circ}{W}{}^{1,0}_p(\Omega_T)$, be the subspace of $W^{1,0}_p(\Omega_T)$ of those functions whose trace on $\partial\Omega(t)$ vanishes for a.e. $t \in [0,T]$. Also let $V_{2,p}(\Omega_T)$ denote the Banach space with norm

(3.5) $$\|u\|_{V_{2,p}(\Omega_T)} = \operatorname*{ess\,sup}_{0<t<T} \|u\|_{2,\Omega(t)} + \|\nabla_x u\|_{p,\Omega_T} .$$

with $\overset{\circ}{V}_{2,p}(\Omega_T)$ we denote the subspace of those functions in $V_{2,p}(\Omega_T)$ whose trace on $\partial\Omega(t)$ is zero for a.e. $t \in [0,T]$. If $p = 2$ we denote $V_{2,2}(\Omega_T)$ and $\overset{\circ}{V}_{2,2}(\Omega_T)$ respectively by $V_2(\Omega_T)$, $\overset{\circ}{V}_2(\Omega_T)$.

Let $V_2^{1,0}(\Omega_T)$ ($\overset{\circ}{V}{}_2^{1,0}(\Omega_T)$) denote the subspace of $W_2^{1,0}(\Omega_T)$ ($\overset{\circ}{W}{}_2^{1,0}(\Omega_T)$) of those functions for which the maps $t \to \|u\|_{2,\Omega(t)}$ are continuous, and with norm defined by (3.5) with $p = 2$ and the "ess" deleted. We will use vector valued versions of these spaces by making the following convention. If $X(\Omega_T)$ is any one of the spaces defined and $\vec{v} : \Omega_T \to \mathbb{R}^2$, by $\vec{v} \in X(\Omega_T)$ we mean that each component of $\vec{v}$ belongs to $X(\Omega_T)$.

Also if $u \in X(Q)$ for every cylindrical domain $Q \subset \Omega_T$ we write $u \in X^{loc}(\Omega_T)$.

The proof of the following embedding lemma can be found in [11].

<u>Lemma 3.1:</u> Let $Q \equiv \Omega' \times (t_1,t_2)$, where $\Omega' \subset \Omega$ and $0 < t_1 < t_2 \le T$, be a cylindrical domain in $\mathbb{R}^{N+1}$. There exists a constant C depending only upon the dimension N, p, and meas $Q = |Q|$ such that if $u \in \overset{\circ}{V}_{2,p}(Q)$, then

$$\|u\|_{r,s,Q} \le C(p,N,|Q|)\|u\|_{\overset{\circ}{V}_{2,p}(Q)} ,$$

where $r,s \ge 1$ are connected by the relation

$$\frac{1}{s} + \frac{2N}{(Np + 2p - 2N)r} = \frac{N}{(Np + 2p - 2N)}$$

and

$$r \in \left[2, \frac{Np}{N-p}\right]; \quad s \in [p,\infty], \quad \text{for} \quad N > p > \frac{2N}{N+2}$$

$$r \in [2,\infty); \quad s \in \left(p - 2 + \frac{2p}{N}, \infty\right], \quad \text{for} \quad 1 < N \le p .$$

We will use the following particular case of Lemma 3.1.

<u>Corollary 3.1.</u> Let $u \in \overset{\circ}{V}_{2,p}(Q)$ and let $N = 2$. Then there exists a constant $C = C(p,|Q|)$ such that

$$\|u\|_{2p,Q} \le C\|u\|_{\overset{\circ}{V}_{2,p}(Q)} .$$

<u>Proof:</u> We take $r = s = p\left(\frac{N+2}{N}\right)$ in Lemma 3.1 and verify that the common value of r and s falls in the admissible range.

Corollary 3.2: Let $Q \subset \Omega_T$ and $u \in V_{2,p}(Q)$, $p > 1$. Then *for every cylindrical domain* Q' such that $\bar{Q}' \subset Q$

$$\|u\|_{2p,Q'} \leq C\|u\|_{V_{2,p}(Q)}$$

where C depends upon p and $\mathrm{dist}(Q,Q')$.

Proof: Q has the form $Q \equiv G \times (t_1,t_2)$, where G is a region in Ω and $0 < t_1 < t_2 \leq T$. Let K be a compact contained in G and set $Q' \leq K \times (t_1,t_2)$. Construct a smooth cutoff function $\varphi \in C_0(G)$ such that $\varphi \equiv 1$ on K. Then $u\varphi \in \mathring{V}_{2,p}(Q)$ and

$$\|u\varphi\|_{2p,Q} \leq C(p,|Q|)\|u\varphi\|_{\mathring{V}_{2,p}(Q)} .$$

The corollary follows from the particular construction of φ.

Next we describe the function spaces where $\vec{v}$ will be found.

Denote with $J(\Omega)$ the closure in the norm of $[L_2(\Omega)]^2$ of $\mathcal{D}_\sigma(\Omega)$, the space of infinitely differentiable, divergence free vector fields $\vec{\psi}$, compactly supported in Ω. Let also $J_1(\Omega)$ denote the closure of $\mathcal{D}_\sigma(\Omega)$ in the norm

$$\|\vec{\psi}\|_{J_1(\Omega)} = \||\nabla_x \vec{\psi}|\|_{2,\Omega} .$$

Set $J(\Omega_T) = L^2(0,T;J(\Omega))$; $J_1(\Omega_T) = L^2(0,T;J_1(\Omega))$, $J_{1,0}(\Omega_T) = C(0,T;J(\Omega)) \cap J_1(\Omega_T)$; $J_{1,1}(\Omega_T) = [W_2^{1,1}(\Omega_T)]^2 \cap J_1(\Omega_T)$; $J_1^\infty(\Omega_T) = L^\infty(0,T;J(\Omega)) \cap J_1(\Omega_T)$. Note that $J_1^\infty(\Omega_T) \subset [\mathring{V}_2(\Omega_T)]^2$, and $J_{1,0}(\Omega_T) \subset [\mathring{V}_2^{1,0}(\Omega_T)]^2$.

4. THE WEAK FORMULATION

Let $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) = 0$. Then multiplying the first of (2.11) by φ and performing (formal) integration by parts, we obtain

$$\tag{4.1} \begin{aligned} &\iint_{\Omega_T} \{-\beta(u)\varphi_t + \nabla_x K(u)\cdot\nabla_x\varphi + \vec{v}\cdot\nabla_x u\varphi\}dxdt = \\ &\quad = -\int_{S_T} g(x,t,u)\varphi d\sigma + \int_{\Omega(0)} \beta(u_0)\varphi(x,0)dx , \end{aligned}$$

where $\beta(\cdot)$ is the maximal monotone graph

$$\beta(s) \equiv \begin{cases} s\,, & s > 0 \\ [-L,\ 0]\,, & s = 0 \\ s - L\,, & s < 0 \end{cases} \tag{4.2}$$

The formal calculations leading to (4.1) are routine and we refer to [7, 12], for details. Here we only remark that the jump of $\beta(\cdot)$ at zero takes into account the interface relation in (2.11).

Since the graph $\beta(\cdot)$ is multivalued, $\beta(u(x,t))$ has to be interpreted as a function $w(x,t) \subset \beta(u(x,t))$, the inclusion being intended in the sense of the graphs. In order to simplify the symbolism we will keep the symbol $\beta(u(x,t))$, bearing in mind the way it has to be interpreted.

Since $u_0 \neq 0$ except on Γ, $\beta(u_0(x))$ is unambigously a.e. defined in $\Omega(0)$.

To obtain a weak formulation of (2.5)', (2.6)-(2.9), consider a smooth, divergence free vector valued function $\vec{\psi}$ which is compactly supported in $\Omega_1(t)$ for all $t \in [0,T]$ and $\vec{\psi}(x,T) \equiv 0$. Take the "dot" product of (2.5)' by $\vec{\psi}$ and integrate by parts in Ω_1. Routine calculations [9, 10], give

$$\iint_{\Omega_1} \{-\vec{v}\cdot\vec{\psi}_t + \nu\nabla_x\vec{v}:\nabla_x\vec{\psi} - \vec{f}(u)\cdot\vec{\psi}\}dxdt = \int_{\Omega_1(0)} \vec{v}_0(x)\vec{\psi}(x,0)dx\ . \tag{4.3}$$

Note that Ω_1 is the set where $u > 0$.

<u>Definition:</u> By a weak solution of (2.11), (2.5)', (2.6)-(2.9), we mean a pair $(u,\vec{v})$ such that

(i) $u \in V_2(\Omega_T) \cap C(\Omega_T)$

(ii) $\vec{v} \in J_1^{\infty}(\Omega_T)$; $\vec{v} = 0$ a.e. on the set $[u < 0] \equiv \{(x,t) \in \Omega_T | u(x,t) < 0\}$.

(iii) u and $\vec{v}$ satisfy

$$\iint_{\Omega_T} \{-\beta(u)\varphi_t + \nabla_x K(u)\cdot\nabla_x\varphi + (\vec{v}\cdot\nabla_x u)\varphi\}dxdt = -\int_{S_T} g(x,t,u)\varphi d\sigma + \int_{\Omega(0)} \beta(u_0)\varphi(x,0)dx \tag{4.4}$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) \equiv 0$, and

$$(4.5)\qquad \iint_{[u>0]} \{-\vec{v}\cdot\vec{\psi}_t + \nu\nabla_x\vec{v}:\nabla_x\vec{\psi} - \vec{f}(u)\vec{\psi}\}dxdt = \int_{\Omega_1(0)} \vec{v}_0(x)\vec{\psi}(x,0)dx$$

for all $\vec{\psi} \in J_{1,1}(\Omega_T)$ such that $\nabla_x\cdot\vec{\psi} = 0$, $\vec{\psi}(x,T) = 0$ and $\text{supp}\,\vec{\psi}(\cdot,t) \subset [u > 0](t)$.

Remarks: (i) Since we require u to be continuous, the set $[u > 0]$ is open in the relative topology of Ω_T, and therefore the last integral identity is well defined.

(ii) The integrals in the identity of the temperature are well defined, modulo some basic assumptions listed below.

5. ASSUMPTIONS AND STATEMENT OF RESULTS

With respect to the data $g(x,t,u)$, $u_0(x)$, $\vec{v}_0(x)$, $\vec{f}(u)$ we assume the following

[A_1] $u_0(x) \neq 0$ a.e. in $\Omega(0)$, $u_0(x) > 0$, $x \in \Omega_1(0)$, $u_0(x) < 0$ on $\Omega_2(0)$. Moreover u_0 is essentially bounded and

$$\|u_0\|_{\infty,\Omega(0)} < K_0$$

where K_0 is a known positive constant.

[A_2] The function $g(x,t,\xi)$ is continuous over $\bar{S}_T \times R$, and satisfies the growth condition

$$|g(x,t,\xi)| < K_0 + K_1|\xi|$$

where K_1 is a known positive constant. Moreover $\xi \to g(x,t,\xi)$ is monotone at the origin for all $(x,t) \in S_T$, i.e. $g(x,t,\xi)\text{sign}\xi \geq 0$.

[A_3] With respect to $K(\cdot)$ we assume that $\xi \to K(\xi)$ is Lipschitz continuous in $R\backslash\{0\}$, it is monotone increasing, and satisfies

$$0 < \lambda_0 < K'(\xi) < \lambda_1, \text{ a.e. } \xi \in R\backslash\{0\} ,$$

where $\lambda_0 = \gamma_0\gamma_1^{-1}$ and $\lambda_1 = \gamma_1\gamma_0^{-1}$.

[A_4] $\vec{v}_0(x) \in J(\Omega)$ and $\vec{v}_0(x) = 0$ a.e. in $\Omega_2(0)$.

[A_5] $\vec{f}$ is Lipschitz continuous in R and

$$|\vec{f}(s_1) - \vec{f}(s_2)| < K_2|s_1 - s_2|$$

for some constant K_2 and all $s_i \in \mathbb{R}$, $i = 1,2$. We will also assume $\vec{f}(0) = 0$. This is no loss of generality since $\vec{f}(0) = 0$ can be always be realized by addition of a conservative force incorporated in the pressure term.

Let $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$, be a multi-index of length $|\alpha| = \alpha_1 + \alpha_2$. Formally with $D_x^\alpha F$ we denote the derivatives of F of the form

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_1^{\alpha_2}} F .$$

Also for $\sigma, \eta \in (0,1)$ let $H_{loc}^{\sigma,\eta}(\Omega_T)$ denote the space of those functions which are Hölder continuous on compact subsets of Ω_T with Hölder exponents σ with respect to the space variables and η with respect to time.

We can now state our main result

Theorem: Under assumptions $[A_1]$-$[A_5]$ problem (2.11), (2.5)', (2.6)-(2.9) has a weak solution. Moreover

(i) $u \in L_\infty(\Omega_T)$

(ii) If $\xi \to K(\xi) \in C^\infty(\mathbb{R}^-)$ then $u \in C^\infty([u < 0])$.

(iii) If $\xi \to K(\xi) \in C^\infty(\mathbb{R}^+)$ and if $\xi \to \vec{f}(\xi) \in C^\infty(\mathbb{R})$ then for every multiindex α

(a)
$$D_x^\alpha u \in L_\infty^{loc}(\Omega_T \cap [u > 0])$$
$$D_x^\alpha u \in H_{loc}^{\sigma,\sigma/2}(\Omega_T \cap [u > 0]), \ \sigma = \sigma(\alpha) \in (0,1) .$$

(b)
$$x \to \vec{v}(x,t) \in C^\infty(\Omega \cap [u(\cdot,t) > 0]), \quad \text{a.e.} \quad t \in [0,T] ,$$
$$D_x^\alpha \vec{v} \in L_\infty^{loc}(\Omega_T \cap [u > 0]), \quad \text{for every} \quad \alpha .$$

Remarks: (i) Assumptions $[A_1]$-$[A_2]$ are somewhat stronger than what is needed for the proof of the theorem and have been formulated in order to simplify the arguments. In fact the monotonicity of $\xi \to g(x,t,\xi)$ at zero can be relaxed, and on $u_0(x)$ one only needs to assume $u_0(x) \in L_2(\Omega(0))$. We will indicate later how this can be done.

(ii) If in (2.11) the variational boundary data are replaced by

$$u|_{S_T} = h(x,t), \qquad (x,t) \in S_T$$

then we have a Dirichlet problem whose weak formulation can be derived in an analogous way. However, it is necessary to take test functions $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$. The proof of the theorem carries over to this situation, modulo the obvious changes due to the different nature of the boundary data. We omit the details for the Dirichlet problem.

6. PROOF OF THE THEOREM

The plan is to obtain the solution $u, \vec{v}$ as a limit of nets $\{u_\varepsilon\}$, $\{\vec{v}_\varepsilon\}$ solutions of certain approximating problems solved in all Ω_T. Since $\vec{v}$ *must act* ultimately only on the set $[u > 0]$, we introduce in the approximating process a penalization acting, roughly speaking, on $[u < 0]$.

Let $\varepsilon > 0$ be fixed and consider the problem of finding a pair $(u_\varepsilon, \vec{v}_\varepsilon)$, $u_\varepsilon \in V_2(\Omega_T)$, $\vec{v} \in J_1^\infty(\Omega_T)$ satisfying

$$\text{(6.1)} \qquad \begin{aligned} &\iint_{\Omega_T} \{-\beta(u_\varepsilon)\varphi_t + \nabla_x K(u_\varepsilon)\cdot\nabla_x\varphi + \vec{v}_\varepsilon\cdot\nabla_x u_\varepsilon \varphi\} dxdt = \\ &= -\int_{S_T} g(x,t,u_\varepsilon)\varphi d\sigma + \int_\Omega \beta(u_0)\varphi(x,0)dx \end{aligned}$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) \equiv 0$, and

$$\text{(6.2)} \qquad \iint_{\Omega_T} \{-\vec{v}_\varepsilon\cdot\vec{\Psi}_t + \nu\nabla_x\vec{v}_\varepsilon : \nabla_x\vec{\Psi} + \varepsilon^{-1}H_\varepsilon(u_\varepsilon)\vec{v}_\varepsilon\vec{\Psi} - \vec{f}(u_\varepsilon)\vec{\Psi}\} dxdt = \int_\Omega \vec{v}_0(x)\vec{\Psi}(x,0)dx\ ,$$

for all $\vec{\Psi} \in J_{1,1}(\Omega_T)$, $\vec{\Psi}(x,T) \equiv 0$. Here $\varepsilon^{-1}H_\varepsilon(\cdot)$ is a penalty term defined by

$$H_\varepsilon(s) = \begin{cases} 1 & , \quad -\infty < s < -2\varepsilon \\ -\varepsilon^{-1}s - 1 & , \quad -2\varepsilon < s < -\varepsilon \\ 0 & , \quad -\varepsilon < s < \infty , \end{cases}$$

and $\vec{v}_0(x)$ here is the extension of $\vec{v}_0(x)$ via $\vec{0}$ into $\Omega_2(0)$.

The crux of the matter lies in the proof of the following facts.

<u>Proposition 6.1:</u> For all $\varepsilon > 0$ the system (6.1)-(6.2) admits a solution $(u_\varepsilon, \vec{v}_\varepsilon)$. Moreover there exist constants C_0, C_1, C_2 depending upon the data in assumptions $[A_1]$-$[A_5]$ but independent of ε such that

(a) $\|u_\varepsilon\|_{V_2(\Omega_T)}, \quad \|\vec{v}_\varepsilon\|_{J_1^\infty(\Omega_T)} < C_0$

(b) $\frac{1}{\varepsilon} \iint_{\Omega_T} H_\varepsilon(u_\varepsilon) |\vec{v}_\varepsilon|^2 dx d\tau < C_1$

(c) $\|u_\varepsilon\|_{\infty,\Omega_T} < C_2$.

For every compact $K \subset \Omega_T$, there exist constants C_3, C_4 depending <u>upon K and ε,</u> such that

(d) $\|\frac{\partial}{\partial t} u_\varepsilon\|_{2,K} < C_3(K,\varepsilon)$

(e) $\|\vec{v}_\varepsilon\|_{\infty,K} < C_4(K,\varepsilon)$.

<u>Proposition 6.2:</u> The net $\{u_\varepsilon\}$ is equicontinuous over Ω_T; that is for every compact $K \subset \Omega_T$, there exists a nondecreasing, continuous function $\omega_K(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_K(0) = 0$, dependent on $\mathrm{dist}(K, \partial\Omega_T)$ but not upon ε, such that

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| < \omega_K(|x_1 - x_2| + |t_1 - t_2|^{1/2})$$

for every pair $(x_i, t_i) \in K$, $i = 1,2$.

By Proposition 6.1, the nets $\{u_\varepsilon\}$, $\{\vec{v}_\varepsilon\}$ are weakly compact in $W_2^{1,0}(\Omega_T)$, and $J_1(\Omega_T)$ respectively. Hence for a subnet relabeled with ε

$$u_\varepsilon \to u \text{ weakly in } W_2^{1,0}(\Omega_T), \text{ and}$$

$$\vec{v}_\varepsilon \to \vec{v} \text{ weakly in } J_1(\Omega_T) .$$

By proposition 6.1(c) and Proposition 6.2, the net $\{u_\varepsilon\}$ is equibounded and equicontinuous on every compact $K \subset \Omega_T$. Therefore a subnet can be selected and relabed with ε such that

$$u_\varepsilon \to u \text{ uniformly on compacts } K \subset \Omega_T$$

and consequently

$$u_\varepsilon \to u \text{ strongly in } L_2(\Omega_T)$$

Because of the equicontinuity of $\{u_\varepsilon\}$ the uniform limit u is continuous in Ω_T, and therefore the set $[u > 0]$ is open in the relative topology of Ω_T.

The proof of these propositions is lengthy and technically involved. We postpone it to the next sections and show here how to conclude the proof of the theorem, using these facts.

[6.A]. <u>The identity of the temperature.</u>

Since $\{u_\varepsilon\}$ is equibounded in Ω_T, from the definition (4.2) of the graph β, it follows that the net $\{\beta(u_\varepsilon)\}$ is also equibounded in Ω_T. Therefore the selection of subnets can be made in such a way as to insure

$$\beta(u_\varepsilon) \to w \text{ weakly in } L_2(\Omega_T) .$$

By monotonicity of $\beta(\cdot)$ we have $w \subset \beta(u)$ in the sense of the graphs. As before we will write $\beta(u)$ instead of w.

By the trace theorem [14], for every $\eta > 0$ there exists a constant $C(\eta)$ such that

$$\int_{S_T} |u_\varepsilon - u|^2 d\sigma \le \eta \|\nabla_x(u_\varepsilon - u)\|^2_{2,\Omega_T} + C(\eta)\|u_\varepsilon - u\|^2_{2,\Omega_T} .$$

Since the norms $\|\nabla_x(u_\varepsilon - u)\|_{2,\Omega_T}$ are equibounded and $u_\varepsilon \to u$ strongly in $L_2(\Omega_T)$, the above implies

$$u_\varepsilon \to u \text{ strongly in } L_2(S_T)$$

and consequently in view of the continuity of $g(x,t,u)$

$$g(x,t,u_\varepsilon) \to g(x,t,u) \quad \text{strongly in } L_2(S_T) .$$

By assumption $[A_3]$, the definition of $K(\cdot)$ and the equicontinuity of $\{u_\varepsilon\}$ over compact subsets of Ω_T it follows that $\{K(u_\varepsilon)\}$ is equicontinuous on compacts $K \subset \Omega_T$ and hence the selection of subnets can be made to include

$$K(u_\varepsilon) \to K(u) \quad \text{uniformly on compacts } K \subset \Omega_T .$$

Consequently

$$K(u_\varepsilon) \to K(u) \quad \text{strongly in } L_2(\Omega_T) .$$

Now we also have

$$\nabla_x K(u_\varepsilon) \to z \quad \text{weakly in } L_2(\Omega_T) .$$

Let $\varphi \in C_0^\infty(\Omega_T)$. Then

$$\iint_{\Omega_T} \nabla_x K(u_\varepsilon)\varphi dxdt = -\iint_{\Omega_T} K(u_\varepsilon)\nabla_x \varphi dxdt$$

and letting $\varepsilon \to 0$

$$\iint_{\Omega_T} z\varphi = -\iint_{\Omega_T} K(u)\nabla_x \varphi dxdt .$$

This implies $z \equiv \nabla_x K(u)$. For the nonlinear term involving $\vec{v}_\varepsilon$ in (6.1) we have,

$$\iint_{\Omega_T} \vec{v}_\varepsilon \cdot \nabla_x u_\varepsilon \varphi dxdt = -\iint_{\Omega_T} \vec{v}_\varepsilon \cdot \nabla_x \varphi u_\varepsilon dxdt \to -\iint_{\Omega_T} \vec{v} \cdot \nabla_x \varphi u dxdt = \iint_{\Omega_T} \vec{v} \cdot \nabla_x u \varphi dxdt ,$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$.

Finally, letting $\varepsilon \to 0$ in (6.1) we obtain the identity (4.4) for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) \equiv 0$.

[6.B] <u>The identity of the velocity</u>

Let K be a compact contained in the open set $[u > 0]$. Since $u_\varepsilon \to u$ uniformly on K, there exists $\varepsilon_0 > 0$ so small that $u_\varepsilon(x,t) > 0$, $\forall (x,t) \in K$ and all $\varepsilon < \varepsilon_0$. Let $\vec{\psi} \in J_{1,1}(\Omega_T)$, $\vec{\psi}(x,T) \equiv 0$ and $\text{supp } \vec{\psi}(\cdot,t) \subset K$, then for the term involving the penalization in (6.2) we have

$$\varepsilon^{-1} \iint_K H_\varepsilon(u_\varepsilon)\vec{v}_\varepsilon \vec{\psi} \, dxdt = 0 \qquad \forall \, \varepsilon < \varepsilon_0 .$$

Letting $\varepsilon \to 0$ in (6.2) with the indicated choice of $\vec{\psi}$, and making use of proposition 6.1, we obtain (4.5). As $K \subset [u > 0]$ was arbitrary, we obtain (4.5) for all such $\vec{\psi}$ with $\text{supp}\,\vec{\psi}(\cdot,t) \subset [u > 0](t)$.

It remains to show that $\vec{v} = 0$ a.e. on $[u < 0]$.

By Proposition 6.1

$$\frac{1}{\varepsilon} \iint_{\Omega_T} H_\varepsilon(u_\varepsilon)|\vec{v}_\varepsilon|^2 dx d\tau \leq C_1 .$$

Let K be a compact contained in $[u < 0]$. From the uniform convergence of u_ε to u, it follows that there exists ε_K so small that $H_\varepsilon(u_\varepsilon) \equiv 1$ on K for all $\varepsilon < \varepsilon_K$.

Therefore

$$\iint_K |\vec{v}_\varepsilon|^2 dx d\tau \leq \varepsilon C_1 .$$

Letting $\varepsilon \to 0$, by lower semicontinuity of the norm we obtain $\vec{v} = 0$ a.e. in K, and since $K \subset [u < 0]$ is arbitrary, $\vec{v} = 0$ a.e. on $[u < 0]$.

[6.C] Regularity

Statement (i) will be proved in Proposition 6.1.

(ii). On the set $[u < 0]$, $\vec{v} = 0$ a.e. Therefore the temperature satisfies

$$\frac{\partial}{\partial t} u - \Delta K(u) = 0 ,$$

in the sense of distributions over $[u < 0]$. Note that here we have used the definition (4.2) of the graph $\beta(\cdot)$. For every cylindrical domain $Q \subset \Omega_T \cap [u < 0]$ we have $u \in V_2^{1,0}(Q)$, Therefore if $K(\cdot)$ is infinitely differentiable on $(-\infty,0)$, the statement is a consequence of classical results [11].

(iii). On the set $[u > 0]$, u, $\vec{v}$ satisfy

$$\frac{\partial}{\partial t} u - \Delta K(u) + \vec{v}\cdot\nabla_x u = 0$$

$$\frac{\partial}{\partial t} \vec{v} - \nu\Delta\vec{v} + \nabla_x p = \vec{f}(u)$$

in the sense of distributions over $[u > 0]$, for a measurable function p. Moreover for every cylindrical domain $Q \subset [u > 0]$ we have $u \in V_2^{1,0}(Q)$, and by Proposition (6.1), $u \in L_\infty(Q)$ and $\vec{v} \in [L_6(Q)]^2$. Hence the stated regularity follows from the results on the local smoothness of weak solutions for the bidimensional Boussinnesq system established in [1].

7. PROOF OF PROPOSITION 6.1

Denote with $\beta_m(\cdot)$ a sequence of smooth functions in $\mathbb{R}$ such that $\beta_m(s) \to \beta(s)$ over compact subsets of $\mathbb{R}\setminus\{0\}$ and satisfying

(i) β_m are monotone increasing

(ii) $\beta_m'(s) \geq 1, \quad \forall s \in \mathbb{R}$

(iii) $\beta_m(s) = s$ for $s > \frac{1}{m}$, and $\beta_m(s) = s - L$ for $s < -\frac{1}{m}$.

Such a sequence can obviously be constructed.

Let also $\{\vec{v}_{0,m}\}$ be a sequence in $J_1(\Omega)$ such that

$$\vec{v}_{0,m} \to \vec{v}_0 \quad \text{in} \quad J(\Omega) .$$

We fix $\varepsilon > 0$ and for each $m \in \mathbb{N}$ consider the following

Auxiliary Problem: Find $u_m \in V_2^{1,0}(\Omega_T)$, $\vec{v}_m \in J_{1,1}(\Omega_T)$ satisfying

$$\iint_{\Omega_T} \{-\beta_m(u_m)\varphi_t + \nabla_x K(u_m)\cdot\nabla_x\varphi + \vec{v}_m\cdot\nabla_x u_m\varphi\}dxd\tau = \tag{7.1}$$

$$= -\int_{S_T} g(x,t,u_m)\varphi d\sigma + \int_\Omega \beta_m(u_0)\varphi(x,0)dx$$

$$\forall \varphi \in W_2^{1,1}(\Omega_T) \quad \text{such that} \quad \varphi(x,T) \equiv 0$$

$$\iint_{\Omega_T} \{-\vec{v}_m\cdot\vec{\psi}_t + \nu\nabla_x\vec{v}_m\cdot\nabla_x\vec{\psi} + \frac{1}{\varepsilon}H_\varepsilon(u_m)\vec{v}_m\cdot\vec{\psi} - \tag{7.2}$$

$$-\vec{f}(u_m)\vec{\psi}\}dxd\tau = \int_\Omega \vec{v}_{0,m}\cdot\vec{\psi}(x,0)dx$$

$$\forall \vec{\psi} \in J_{1,1}(\Omega_T) \quad \text{such that} \quad \vec{\psi}(x,T) \equiv 0 .$$

We make the convention of denoting with C a generic positive constant depending upon quantities that will be specified as the constant appears.

Proposition 7.1: For each $m \in \mathbb{N}$ the auxiliary problem (7.1)-(7.2) has a solution. Moreover there exist constants C depending upon the data in assumptions $[A_1]$-$[A_5]$, but independent of m and ϵ, such that

$$(7.3) \qquad \|u_m\|_{V_2(\Omega_T)}, \quad \|\vec{v}_m\|_{J_1^\infty(\Omega_T)} \le C$$

$$(7.4) \qquad \frac{1}{\epsilon} \iint_{\Omega_T} H_\epsilon(u_m)|\vec{v}_m|^2 dx d\tau \le C .$$

Proof of Proposition 7.1: We employ a Galerkin procedure. In $L_2(\Omega)$, introduce the orthonormal basis $\{z_i(x)\}$ generated by the problems

$$-\Delta z_i = \lambda_i z_i \qquad \text{in } \Omega$$

$$\frac{\partial z_i}{\partial \vec{n}} = 0 \qquad \text{on } \partial\Omega$$

In $J(\Omega)$ we introduce the orthonormal basis generated by the Stokes problems

$$-\Delta\vec{\zeta}_i + \nabla P = \mu_i \zeta_i \quad \text{in } \Omega$$

$$\nabla\cdot\vec{\zeta}_i = 0$$

$$\vec{\zeta}_i\big|_{\partial\Omega} = 0$$

where P is a scalar function representing a pressure. From [10,13] it follows that $\{\vec{\zeta}_i\}$ form a complete orthonormal set in $J(\Omega)$. By Hilbert-Schmidt theorem [13] we see that any smooth divergence free vector valued function $\vec{\eta}$, compactly supported in Ω has an absolutely and uniform convergent representation

$$\vec{\eta}(x) = \sum_{i=1}^{\infty} (\vec{\eta},\vec{\zeta}_i)_{L_2(\Omega)}\vec{\zeta}_i(x) .$$

Moreover the derivatives of $\vec{\eta}(x)$ have an absolutely and uniformly convergent representation obtained by term by term differentiation.

We represent the initial data $u_0(x)$, $\vec{v}_{0,m}(x)$ as

$$u_0(x) = \sum_{i=1}^{\infty} c_i^0 z_i(x)$$

$$\vec{v}_{0,m}(x) = \sum_{i=1}^{\infty} d_i^0 \vec{\zeta}_i(x) \, .$$

For $\ell \in \mathbb{N}$ fixed set

$$\vec{v}_\ell^*(x,t) = \sum_{i=1}^{\ell} d_i^*(t)\vec{\zeta}_i(x) \tag{7.5}$$

and denote by $W_{m,\ell}(x,t)$ the unique solution of

$$\frac{\partial}{\partial t} W_{m,\ell} - \Delta K(\beta_m^{-1}(W_{m,\ell})) + \vec{v}_\ell^* \cdot \nabla_x \beta_m^{-1}(W_{m,\ell}) = 0 \tag{7.6}$$

$$-\nabla_x K(\beta_m^{-1}(W_{m,\ell})) \cdot \vec{N} = g(x,t,\beta_m^{-1}(W_{m,\ell})) \quad \text{on } S_T \tag{7.7}$$

$$W_{m,\ell}(x,0) = V_\ell(x) \, , \tag{7.8}$$

in the sense of projections over the span of $\{z_1, z_2, \ldots, z_\ell\}$, where

$$V_\ell(x) = \sum_{i=1}^{\ell} c_{m,i} z_i(x)$$

and

$$\beta_m(u_0(x)) = \sum_{i \geq 1} c_{m,i} z_i(x) \, .$$

Namely denoting with P_ℓ the $L_2(\Omega)$ projection onto the linear span of $\{z_1, z_2, \ldots, z_\ell\}$, $W_{m,\ell}$ is the unique element in $W_2^{1,1}(\Omega_T)$ satisfying (7.8) and

$$\begin{aligned} \int_\Omega \Big\{ & \frac{\partial}{\partial t} W_{m,\ell} P_\ell \varphi + \nabla_x K(\beta_m^{-1}(W_{m,\ell})) \cdot \nabla_x P_\ell \varphi + \\ & + \vec{v}_\ell^* \cdot \nabla_x \beta_m^{-1}(W_{m,\ell}) P_\ell \varphi \Big\} dx dt = -\int_{S_T} g(x,t,\beta_m^{-1}(W_{m,\ell})) P_\ell \varphi d\sigma \end{aligned} \tag{7.9}$$

for all $\varphi \in W_2^1(\Omega)$, and all $t \in [0,T]$.

For the construction of $W_{m,\ell}$ (global in time), we refer to [2,3,4].

We will employ the function $W_{m,\ell}$ so obtained to construct the function $\vec{v}_\ell$ solution of

$$\frac{\partial}{\partial t}\vec{v}_\ell - \nu\Delta\vec{v}_\ell + \varepsilon^{-1}H_\varepsilon(\beta_m^{-1}(W_{m,\ell}))\vec{v}_\ell = \vec{f}(\beta_m^{-1}(W_{m,\ell})) \quad \text{in } \Omega_T \tag{7.10}$$

$$\vec{v}_\ell(x,t) = 0 \qquad (x,t) \in S_T \tag{7.11}$$

$$\vec{v}_\ell(x,0) = \sum_{i=1}^{\ell} d_i^0\vec{\zeta}_i(x) , \tag{7.12}$$

in the sense of projections over the span of $\{\vec{\zeta}_1,\vec{\zeta}_2,\ldots,\vec{\zeta}_\ell\}$. Namely denoting with Π_ℓ the $J(\Omega)$ projection onto the linear span of $\{\vec{\zeta}_1,\vec{\zeta}_2,\ldots,\vec{\zeta}_\ell\}$, $\vec{v}_\ell$ is the unique element in $J_{1,1}(\Omega_T)$ satisfying (7.12) and

$$\begin{aligned}\int_\Omega \Big\{\frac{\partial}{\partial t}\vec{v}_\ell\, \Pi_\ell\vec{\psi} + \nu\nabla_x\vec{v}_\ell : \nabla_x\Pi_\ell\vec{\psi} + \\ + \varepsilon^{-1}H_\varepsilon(\beta_m^{-1}(W_{m,\ell}))\vec{v}_\ell\cdot\Pi_\ell\vec{\psi}\Big\}dx = \int_\Omega \vec{f}(\beta_m^{-1}(W_{m,\ell}))\Pi_\ell\vec{\psi}\end{aligned} \tag{7.13}$$

for all $\vec{\psi} \in J_1(\Omega)$ and for all $t \in [0,T]$.

We omit the proof of the existence and uniqueness of $\vec{v}_\ell$ satisfying (7.10) since the construction follows by straightforward modification of standard techniques [10,9].

Denote by $L_\infty[0,T]$ the space of essentially bounded real valued functions in $[0,T]$, and with $H^1[0,T]$ the space of those square summable real valued functions $t \to d(t)$, whose weak derivative $d'(t)$ is square summable over $[0,T]$.

For $\ell \in \mathbb{N}$ set

$X_\ell = (L_\infty[0,T])^\ell$, the cartesian product of $L_\infty[0,T]$ by itself ℓ times

$Y_\ell = (H^1[0,T])^\ell$, the cartesian product of $H^1[0,T]$ by itself ℓ times.

In X_ℓ and Y_ℓ introduce the norms

$$\|d_1(\cdot),d_2(\cdot),\ldots,d_\ell(\cdot)\|^2_{X_\ell} = \operatorname*{ess\,sup}_{0\le t\le T} \sum_{i=1}^{\ell} [d_i(t)]^2$$

$$\|d_1(\cdot),d_2(\cdot),\ldots,d_\ell(\cdot)\|^2_{Y_\ell} = \sum_{i=1}^{\ell} \|d_i\|^2_{2,[0,T]} + \sum_{i=1}^{\ell} \|d_i'\|^2_{2,[0,T]} .$$

The procedure described above defines a map $F_\ell : X_\ell \to Y_\ell$ as follows. Given an ℓ-tuple $\{d_1^*(t),d_2^*(t),\ldots,d_\ell^*(t)\} \in X_\ell$ we construct $W_{m,\ell}(x,t)$ the unique solution of (7.9) with $\vec{v}_\ell^*$ given by (7.5). Then we associate with $\{d_1^*(t),d_2^*(t),\ldots,d_\ell^*(t)\}$, the *Fourier coefficients* $\{d_1(t),d_2(t),\ldots,d_\ell(t)\}$ *of the solution* $\vec{v}_\ell(x,t)$ *of* (7.13).

We need to show that for each $\ell \in \mathbb{N}$, F_ℓ possesses a fixed point in X_ℓ.

This can be done by using the Leray-Schauder fixed point theorem [10]. For this we have to prove that

(i) F_ℓ maps a bounded set B in X_ℓ into itself

(ii) $F_\ell : B \to B$ is compact

(iii) The set of solutions of $\lambda F_\ell(x) = x$, $\lambda \in [0,1]$ is bounded independently of λ.

<u>Lemma 7.1:</u> There exists a constant C which depends only upon $u_0(x)$, Ω_T, L, the constants in assumptions $[A_1]$-$[A_5]$ and which is independent of ε, m, ℓ, $\vec{v}_\ell^*$ such that

$$\|W_{m,\ell}(\cdot,t)\|_{2,\Omega} \le C, \quad \text{all} \quad t \in [0,T] .$$

Moreover setting $\beta_m^{-1}(W_{m,\ell}) = U_{m,\ell}$, there exists a constant C dependent on the previous quantities and independent of ε, m, ℓ, $\vec{v}_\ell^*$ such that

$$\|U_{m,\ell}(\cdot,t)\|^2_{2,\Omega} + \int_0^t \|\nabla_x U_{m,\ell}(\cdot,\tau)\|^2_{2,\Omega} d\tau \le C; \quad t \in [0,T] . \tag{7.14}$$

<u>Proof of Lemma 7.1:</u> The proof is exactly the same as the proof of Lemmas 1 and 2 of [4]. Here we only show why the constants C are independent of $\vec{v}_\ell^*$. The lemma is proved by setting in (7.9)

$$P_\ell \varphi = W_{m,\ell}$$

and performing estimates in the identity so obtained. For the term involving $\vec{v}^*_\ell$, we have

$$\int_\Omega \vec{v}^*_\ell \beta_m^{-1}(W_{m,\ell})\nabla_x W_{m,\ell}dx = \int_\Omega \vec{v}^*_\ell\{\nabla_x \int_0^{W_{m,\ell}} \beta_m^{-1}(\xi)d\xi\}dx =$$

$$= -\int_\Omega [\int_0^{W_{m,\ell}} \beta_m^{-1}(\xi)d\xi]\mathrm{div}\, \vec{v}^*_\ell dx = 0 .$$

__Lemma 7.2:__ (a) There exists a constant C which depends upon the constants in Lemma 7.1, the Lipschitz constant of $\vec{f}(\cdot)$, Ω_T, $\vec{v}_0(x)$ and which is independent of ε, m, ℓ, $\vec{v}^*_\ell$, such that

(7.15) $$\|\vec{v}_\ell(\cdot,t)\|^2_{2,\Omega} + \int_0^t \|\nabla_x\vec{v}_\ell(\cdot,\tau)\|^2_{2,\Omega}d\tau \le C; \quad t \in [0,T]$$

(7.16) $$\frac{1}{\varepsilon}\iint_{\Omega_T} H_\varepsilon(U_{m,\ell})|\vec{v}_\ell|^2 dxd\tau \le C$$

(b) There exists a constant C which depends on m, ε, $\|\vec{v}_{0,m}\|_{J_1(\Omega)}$, but which is independent of ℓ such that

(7.17) $$\|\frac{\partial}{\partial t}\vec{v}_\ell\|_{2,\Omega_T} \le C(m,\varepsilon) .$$

__Proof of Lemma 7.2:__ To prove (7.15) and (7.16) choose $\Pi_\ell\vec{\psi} = \vec{v}_\ell$ in (7.13). To prove (7.17) choose $\Pi_\ell\vec{\psi} = \frac{\partial}{\partial t}\vec{v}_\ell$. The lemma now follows from routine calculations and Gronwall's inequality.

Now (7.15) implies

$$\sum_{i=1}^{\ell} [d_i(t)]^2 \le C, \quad \text{for all } t \in [0,T] ,$$

and therefore if $\{d^*_1(t),d^*_2(t),\dots,d^*_\ell(t)\}$ belongs to the ball of radius $C^{1/2}$ in X_ℓ we have

$$F_\ell\{d^*_1,d^*_2,\dots,d^*_\ell\} \in \text{the ball of radius } C^{1/2} \text{ in } X_\ell .$$

In fact by (7.17) $\{d_1,d_2,\dots,d_\ell\} = F_\ell\{d^*_1,d^*_2,\dots,d^*_\ell\} \in Y_\ell$ and Y_ℓ is compactly embedded in X_ℓ. Therefore F_ℓ is compact. The continuity of F_ℓ is demonstrated by a standard difference argument, and condition (iii) is trivial (see [5] for similar arguments).

Hence we conclude that F_ℓ admits a fixed point in X_ℓ and actually every fixed point lies in Y_ℓ. Let $\vec{v}_\ell$ be a fixed point of F_ℓ and set

$$U_{m,\ell} = \beta_m^{-1}(W_{m,\ell}) .$$

Then $W_{m,\ell}$, $U_{m,\ell}$, $\vec{v}_\ell$ satisfy the identities

$$(7.18) \quad \iint_{\Omega_T} \{\frac{\partial}{\partial t} W_{m,\ell} P_s \varphi + \nabla_x K(U_{m,\ell}) \nabla_x P_s \varphi + \vec{v}_\ell \cdot \nabla_x U_{m,\ell} P_s \varphi\} dxdt = -\int_{S_T} g(x,t,U_{m,\ell}) P_s \varphi d\sigma$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) \equiv 0$,

$$(7.19) \quad \begin{aligned} &\iint_{\Omega_T} \{\frac{\partial}{\partial t} \vec{v}_\ell \cdot \Pi_s \vec{\psi} + \nu \nabla_x \vec{v}_\ell : \nabla_x \Pi_s \vec{\psi} + \\ &+ \varepsilon^{-1} H_\varepsilon(U_{m,\ell}) \vec{v}_\ell \Pi_s \vec{\psi}\} dxdt = \iint_{\Omega_T} \vec{f}(U_{m,\ell}) \cdot \Pi_s \vec{\psi} dxdt \end{aligned}$$

for all $\vec{\psi} \in J_{1,1}(\Omega_T)$ such that $\vec{\psi}(x,T) \equiv 0$. Here s is a positive integer $< \ell$. Now we let $\ell \to \infty$ while $m \in \mathbb{N}$ remains fixed.

The limit process of $\ell \to \infty$ in (7.18) is carried out exactly as in [4]. We remark that a crucial fact in this connection is to show that

$$(7.20) \quad \begin{aligned} &U_{m,\ell} \to u_m \quad \text{strongly in } L_2(\Omega_T) \\ &W_{m,\ell} \to w_m \in \beta_m(u_m), \quad \text{weakly in } L_2(\Omega_T) \end{aligned}$$

These facts were shown in [4] to which we refer for details.

The passage to the limit in (7.19) follows the arguments of [9,10], by making use of the information (7.20).

Note that by lower semicontinuity of the norm the estimates (7.14), (7.15), (7.16), (7.17) are still valid in the limit. Proposition 7.1 is proved.

<u>7-(a) Regularity of</u> u_m <u>and</u> $\vec{v}_m$

Next we give an equivalent formulation of (7.1), by using the Steklov averagings of a function $F \in V_2(\Omega_T)$, defined by

$$F_h(x,t) = \begin{cases} \frac{1}{h}\int_t^{t+h} F(x,\tau)d\tau, & 0 < t < T - h \\ 0 & t > T - h \end{cases}$$

In [11] it is shown that (7.1) is equivalent to

$$(7.21) \quad \iint_{\Omega_t} \{\frac{\partial}{\partial t} [\beta_m(u_m)]_h\varphi + \nabla_x[K(u_m)]_h\cdot\nabla_x\varphi + [\vec{v}_m\cdot\nabla_x u_m]_h\varphi\}dxd\tau = \int_{S_t} [g(x,t,u_m)]_h\varphi d\sigma$$

for all $0 < t < T - h$,

and

$$\beta_m(u_m(x,0)) = \beta_m(u_0(x)) ,$$

for all $\varphi \in W_2^{1,0}(\Omega_T)$.

Lemma 7.3: There exists a constant C independent of ε, such that

$$\operatorname*{ess\,sup}_{\Omega_T} |u_m| \le C \qquad m = 1,2,\ldots .$$

Proof: Let k be a positive real number such that $k > \max\{\frac{1}{m}, \|\beta_m(u_0)\|_{\infty,\Omega}\}$, and consider the function

$$\varphi = ([\beta_m(u_m)]_h - k)^+ = \max\{[\beta_m(u_m)]_h - k, 0\} .$$

It is immediate to verify that $\varphi \in W_2^{1,1}(\Omega_T)$ and therefore it can be used as a test function in (7.21). We obtain

$$(7.21) \quad \begin{aligned} &\int_0^t \int_\Omega \{\frac{1}{2}\frac{\partial}{\partial t}[([\beta_m(u_m)]_h - k)^+]^2 + \nabla_x[K(u_m)]_h\cdot\nabla_x([\beta_m(u_m)]_h - k)^+ + \\ &+ [\vec{v}_m\cdot\nabla_x u_m]_h([\beta_m(u_m)]_h - k]^+\}dxd\tau = -\int_{S_t} [g(x,\tau,u_m)]_h(\beta_m(u_m) - k)^+d\sigma \end{aligned}$$

We perform an integration by parts in the first integral and let $h \to 0$, exploiting the fact that u_m (and hence $\beta_m(u_m)$) belongs to $V_2^{1,0}(\Omega_T)$ [11]. This gives

$$\frac{1}{2}\int_{\Omega(t)}(\beta_m(u_m) - k)^{+2}dx + \iint_{\Omega_t} \nabla_x K(u_m)\cdot\nabla_x(\beta_m(u_m) - k)^+ dxd\tau +$$

$$+ \iint_{\Omega_t} \vec{v}_m\cdot\nabla_x u_m(\beta_m(u_m) - k)^+ dxd\tau = -\int_{S_t} g(x,t,u_m)(\beta_m(u_m) - k)^+ d\sigma \tag{7.22}$$

$$+ \frac{1}{2}\int_{\Omega}(\beta_m(u_0(x)) - k)^{+2}dx .$$

By our choice of k, the last integral in (7.22) is zero. We treat the remaining terms as follows. First observe that by our construction of $\beta_m(\cdot)$ we have

$$\beta_m(s) = s \quad \text{for} \quad s > \frac{1}{m} .$$

Therefore since $k > \frac{1}{m}$ we have $(\beta_m(u_m) - k)^+ = (u_m - k)^+$. From this and routine calculation it follows

$$\iint_{\Omega_t} \nabla_x K(u_m)\cdot\nabla_x(\beta_m(u_m) - k)^+ dxd\tau = \iint_{\Omega_t} K'(u_m)|\nabla_x(u_m - k)^+|^2 dxd\tau \geq 0 .$$

For the term involving the velocity we have

$$\iint_{\Omega_t} \vec{v}_m\cdot\nabla_x u_m(\beta_m(u_m) - k)^+ dxd\tau = \iint_{\Omega_t} \vec{v}_m\cdot\nabla_x(u_m - k)^+(u_m - k)^+ dxd\tau =$$

$$= \frac{1}{2}\iint_{\Omega_t} \vec{v}_m\cdot\nabla_x[(u_m - k)^+]^2 dxd\tau = 0$$

since $\operatorname{div} \vec{v}_m = 0$.

By monotonicity of $g(x,t,u_m)$ at the origin

$$g(x,t,u_m)(u_m - k)^+ \geq 0 .$$

Carrying these estimates in (7.22) and dropping the non-negative terms we obtain

$$\int_{\Omega(t)} (u_m - k)^{+2}dx \leq 0 \quad \text{a.e.} \quad t \in [0,T].$$

This implies $u_m(x,t) \leq k$, a.e. $(x,t) \in \Omega_T$. The bound from below is derived analogously. By taking

$$k = \max\{1, \sup_{m \in \mathbb{N}} \|\beta_m(u_0)\|_\infty\} ,$$

the constant in Lemma 7.3 is made independent of m.

This result will be employed to prove the following lemma

__Lemma 7.4:__ Let K be a compact of Ω_T. Then there exists a constant C depending upon $\mathrm{dist}(K,\partial\Omega_T)$ and ε but not upon m, such that

$$\|\vec{v}_m\|_{\infty,K} \leq C(\varepsilon,K), \qquad m = 1,2,\ldots .$$

__Remark:__ Lemma 7.4 says that the sequence $\{\vec{v}_m\}$ is uniformly bounded on compacts of Ω_T. We stress the fact that the bound does not depend on m, but depends upon the size of the penalty term $\varepsilon^{-1}H_\varepsilon(u_m)$ (and hence upon ε). Also there is no claim of uniform boundedness of $\vec{v}_m$ over all Ω_T, but only on compacts $K \subset \Omega_T$.

For each $m \in \mathbb{N}$ we define the vorticity ω_m of $\vec{v}_m$ as the skew-symmetric tensor (ω_m^{ij}) of entries

$$\omega_m^{ij} = \frac{\partial v^{(i)}}{\partial x_j} - \frac{\partial v^{(j)}}{\partial x_i} , \qquad i,j = 1,2 .$$

Let G, G', G'' be regions in Ω such that $\overline{G''} \subset G'$, $\overline{G'} \subset G$, $\overline{G} \subset \Omega$, and consider the cylindrical domains

$$Q \equiv G \times [t_1,t_2]; \quad Q' \equiv G' \times [t_1',t_2]; \quad Q'' \equiv G'' \times [t_1'',t_2]$$

where $0 < t_1 < t_1' < t_1'' < t_2 \leq T$.

The following local representation of $\vec{v}_m$, will play a role in what follows.

__Lemma 7.5:__ Let $x \to \zeta(x) \in C_0^\infty(G)$ such that $\zeta(x) \equiv 1$ on G'. Then

$$\zeta(x)\vec{v}_m(x,t) = \int_G \zeta(y)\nabla_y H(x-y) \wedge \omega_m(y,t)dy + \vec{A}_m(x,t) , \tag{7.23}$$

where $H(\cdot)$ is the fundamental solution of the Laplace equation and for $x \in G''$, $\vec{A}_m(\cdot,\cdot)$ is harmonic in x and $L_\infty^{loc}(0,T)$, uniformly in m.

<u>Proof of Lemma 7.5:</u> Assume first that $\vec{v}_m \in C^\infty(\Omega_T)$. Then from $-\Delta H(x-y) = \delta_x$ in $\mathcal{D}'(\mathbb{R}^2)$, where δ_x denotes the Dirac mass concentrated at x, we obtain

$$\zeta(x)v_m^{(i)}(x,t) = \int_{\mathbb{R}^2} \nabla_y H(x-y)\cdot\nabla_y[\zeta(y)v_m^{(i)}(y,t)]dy =$$

$$\sum_{j=1}^{2}\int_G \zeta(y)\frac{\partial}{\partial y_j}H(x-y)\frac{\partial}{\partial y_j}v_m^{(i)}(y,t)dy + \int_G [\nabla H(x-y)\cdot\nabla\zeta(y)]v_m^{(i)}(y,t)dy$$

$$= \int_G \zeta(y)\nabla_y H(x-y)\wedge\omega_m(y,t)dy + A_m^{(i)}(x,t) ,$$

where

$$A_m^{(i)}(x,t) = \int_G [\nabla H(x-y)\cdot\nabla\zeta(y)]v_m^{(i)}(y,t) - \int_G \frac{\partial}{\partial y_i}H(x-y)\nabla\zeta(y)\cdot\vec{v}_m(y,t)dy .$$

By a density argument this representation holds for $\vec{v}_m \in J_1^\infty(\Omega_T)$. Because of the choice of the cutoff function ζ, $\vec{A}_m$ is harmonic in $G"$ and $L_\infty^{loc}(0,T)$ uniformly in m.

<u>Remark:</u> Such a representation is similar to Lemma 2 of [18]. The point here is to point out that since $\vec{A}_m$ is harmonic in $x \in G"$ and $L_\infty^{loc}(0,T)$ uniformly in m, we have

$$D_x^\alpha \vec{A}_m \in [L_\infty^{loc}(\Omega_T)]^2, \text{ uniformly in } m .$$

The bounds will depend on dist$(G',G")$, the constant in (7.3), and the multi-index α.

<u>Proof of Lemma 7.4:</u> Since $\vec{v}_m$ satisfies (7.2), denoting with $\vec{v}_{m,h}$ a mollification of $\vec{v}_m$, there exists a differentiable function $p_{m,h}$ such that

$$\frac{\partial}{\partial t}\vec{v}_{m,h} - \nu\Delta\vec{v}_{m,h} = -\nabla_x p_{m,h} + [\vec{f}(u_m) - \varepsilon^{-1}H_\varepsilon(u_m)\vec{v}_m]_h \tag{7.24}$$

in Ω_T (see [18]).

By taking the curl

$$\frac{\partial}{\partial t}\omega_{m,h} - \nu\Delta\omega_{m,h} = \text{curl}[\vec{f}(u_m) - \varepsilon^{-1}H_\varepsilon(u_m)\vec{v}_m]_h \text{ in } \Omega_T . \tag{7.25}$$

We already know that $u_m \in L_\infty(\Omega_T)$ uniformly in m, and that $\vec{v}_m \in J_1^\infty(\Omega_T)$ uniformly in m. Consequently setting

$$\vec{\phi}_m = \vec{f}(u_m) - \varepsilon^{-1}H_\varepsilon(u_m)\vec{v}_m ,$$

we have

$$\phi_{m,h} \in L_2(\Omega_T) \quad \text{uniformly in } m \text{ and } h,$$

and there exists a constant $C(\varepsilon)$ depending upon ess sup$|u_m|$, the Lipschitz constant K_2 of $\vec{f}(\cdot)$ in $[A_5]$, Ω_T, $\vec{v}_0(x)$ and ε such that

$$\|\phi_{m,h}\|_{2,\Omega_T} \le C(\varepsilon), \quad \forall\, m,h$$

Construct a cutoff function $(x,t) \to \zeta(x,t) \in C^\infty(Q)$, such that $\zeta(x,t) \equiv 1$ on Q', $\zeta(x,t_1) \equiv 0$, $x \to \zeta(x,t) \in C_0^\infty(G)$ and $0 \le \zeta \le 1$. Then $\omega_{m,h}\zeta^2 \in C^\infty(Q)$ and vanishes on the parabolic boundary of Q. Multiplying (7.24) by $\omega_{m,h}\zeta^2$ and integrating over Q, after standard calculations we obtain

$$\text{(7.26)} \qquad \operatorname*{ess\,sup}_{t_1 \le t \le t_2} \|\omega_{m,h}\zeta\|^2_{2,G\times\{t\}} + \||\nabla_x\omega_{m,h}|\zeta\|^2_{2,Q} \le C(\|\phi_{m,h}\|^2_{2,\Omega_T} + \|\omega_{m,h}\|^2_{2,\Omega_T}) ,$$

for a constant C, depending upon ε, dist(Q,Q'), the data but independent of m and h. Since $\vec{v}_m \in \mathring{J}_1^\infty(\Omega_T)$ uniformly in m we have that $\omega_{m,h} \in L_2(\Omega_T)$ uniformly in m and h. Therefore from (7.26) recalling the definition of the cutoff function ζ, we deduce that there exists a constant $C(\varepsilon)$ depending upon ε and the data but independent of m and h such that

$$\text{(7.27)} \qquad \|\omega_{m,h}\|_{V_2(Q)} \le C(\varepsilon) .$$

Corollary (3.2), with $p = 2$ implies that $\omega_{m,h} \in L_4(Q'')$ uniformly in m and h. Therefore since the choices of Q, Q', Q'' are arbitrary, we deduce that $\omega_m \in L_4^{loc}(\Omega_T)$, uniformly in m, with bounds depending upon ε.

From the representation (7.23) and the Calderon-Zygmund theory of singular integrals [20], we deduce that

$$\nabla_x\vec{v}_m \in L_4^{loc}(\Omega_T) \quad \text{uniformly in } m ,$$

and therefore $\vec{v}_m \in V_{2,4}^{loc}(\Omega_T)$ uniformly in m. By Corollary 3.2, with $p = 4$ we find $\vec{v}_m \in L_8^{loc}(\Omega_T)$ uniformly in m. Now letting $h \to 0$ in $\mathcal{D}'(\Omega_T)$ in (7.25) we have

$$\frac{\partial}{\partial t} \omega_m - \nu\Delta\omega_m = \text{curl}(\Phi_m) \quad \text{in } \mathcal{D}'(\Omega_T) . \tag{7.28}$$

From (7.27) we see that ω_m is a solution of (7.28) which belongs to $V_2^{loc}(\Omega_T)$. Since $\Phi_m \in L_8^{loc}(\Omega_T)$ uniformly in m, standard parabolic theory [11, 16, 17], implies that $\omega_m \in L_\infty^{loc}(\Omega_T)$ uniformly in m, with local bounds depending upon ε.

The lemma is now a consequence of the representation (7.23).

Finally we employ Lemma 7.4 to show that $\frac{\partial}{\partial t} u_m \in L_2^{loc}(\Omega_T)$.

__Lemma 7.6:__ Let K be a compact of Ω_T. Then there exists a constant C depending upon $\text{dist}(K,\partial\Omega_T)$, ε and the data, but independent of m such that

$$\left\| \frac{\partial}{\partial t} u_m \right\|_{L_2(K)} \le C .$$

__Proof of Lemma 7.6:__ Let $K_1 \subset K_2 \subset \Omega$ be compacts and consider the cylindrical domains

$$Q_1 \equiv K_1 \times [t_1,t_2]; \quad Q_2 \equiv K_2 \times [t_1',t_2'] \quad \text{where}$$

$$0 < t_1' < t_1 < t_2 < t_2' < T .$$

Construct a cutoff function $\varphi(x,t) \in C^\infty(Q_2)$ such that

(i) $\text{supp}\, \varphi \subset Q_2$

(ii) $\varphi \equiv 1, \quad (x,t) \in Q_1$.

Next consider identity (7.21) and set

$$K(u_m) = z_m$$

and

$$\gamma_m(z_m) = \beta_m(K^{-1}(z_m)) .$$

It is clear that by virtue of the assumptions on $K(\cdot)$ it will be sufficient to prove

$$\left\| \frac{\partial}{\partial t} z_m \right\|_{2,K} \le C .$$

In (7.21) with the indicated change of variable, choose the test function

$$\{\frac{\partial}{\partial t}[z_m]_h\}\varphi^2 .$$

Since $\operatorname{supp}\varphi \subset Q_2$, the term involving integrations on S_T drops out, and we have

$$\tag{7.29} \begin{aligned} \iint_{Q_2} \{\frac{\partial}{\partial t}\,[\gamma_m(z_m)]_h \frac{\partial}{\partial t}\,[z_m]_h\varphi^2 + \nabla_x[z_m]_h \frac{\partial}{\partial t}\,\nabla_x[z_m]_h\varphi^2 + \\ + \nabla_x[z_m]_h \frac{\partial}{\partial t}\,[z_m]_h\cdot\nabla_x\varphi^2 + [\vec{v}_m\cdot\nabla_x K^{-1}(z_m)]_h \frac{\partial}{\partial t}\,[z_m]_h\varphi^2\}dxdt = 0 \end{aligned}$$

Consider the first integrand in (7.29) and recall that by virtue of assumption $[A_3]$ and the construction of the sequence $\beta_m(\cdot)$ we have

$$\gamma_m'(s) = \beta_m'[K^{-1}(s)]\cdot K^{-1'}(s) \geq \lambda_1^{-1} \qquad s \in R\backslash\{0\} .$$

Now from the definition of Steklov averaging it follows that

$$\begin{aligned} \frac{\partial}{\partial t}\,[\gamma_m(z_m)]_h \frac{\partial}{\partial t}\,[z_m]_h &= \frac{\gamma_m(z_m(t+h)) - \gamma_m(z_m(t))}{h} \cdot \frac{z_m(t+h) - z_m(t)}{h} \\ &\geq \lambda_1^{-1}\Big[\frac{z_m(t+h) - z_m(t)}{h}\Big]^2 = \lambda_1^{-1}\Big|\frac{\partial}{\partial t}\,[z_m]_h\Big|^2 \end{aligned}$$

For the term involving gradients we have

$$\begin{aligned} \iint_{Q_2} \nabla_x[z_m]_h \frac{\partial}{\partial t}\,\nabla_x[z_m]_h\varphi^2 dxdt &= \frac{1}{2}\iint_{Q_2} \frac{\partial}{\partial t}\,|\nabla_x[z_m]_h|^2\varphi^2 dxdt = \\ &= -\frac{1}{2}\iint_{Q_2} |\nabla_x[z_m]_h|^2\,\frac{\partial\varphi^2}{\partial t}\,dxdt . \end{aligned}$$

These remarks in (7.29) give

$$\lambda_1^{-1} \iint_{Q_2} |\frac{\partial}{\partial t} [z_m]_h|^2 \varphi^2 dxdt = \frac{1}{2} \iint_{Q_2} |\nabla_x [z_m]_h|^2 \frac{\partial \varphi^2}{\partial t} dxdt - 2 \iint_{Q_2} \nabla_x [z_m]_h \frac{\partial}{\partial t} [z_m]_h \varphi \nabla_x \varphi$$

(7.30)

$$- \iint_{Q_2} [\vec{v}_m \cdot \nabla_x K^{-1}(z_m)]_h (\frac{\partial}{\partial t} [z_m]_h) \varphi^2 dxdt = I_1 + I_2 + I_3 .$$

We estimate the I_i, $i = 1,2,3$ separately. From (7.3) it follows that the integral I_1 is uniformly bounded with respect to m and that the bound will depend on $\frac{\partial \varphi}{\partial t}$, i.e. the distance between Q_1 and Q_2.

As for I_2 we use the Cauchy inequality $ab < \eta a^2 + \frac{1}{\eta} b^2$ $\eta > 0$, to obtain

$$|I_2| < \eta \iint_{Q_2} |\frac{\partial}{\partial t} [z_m]_h|^2 \varphi^2 + \frac{4}{\eta} \iint_{Q_2} |\nabla_x [z_m]_h|^2 |\nabla_x \varphi|^2 dxdt$$

We do the same with I_3

$$|J_3| < \eta \iint_{Q_2} |\frac{\partial}{\partial t} [z_m]_h|^2 \varphi^2 dxdt + \frac{1}{\eta} \iint_{Q_2} |[\vec{v}_m \cdot \nabla_x K^{-1}(z_m)]_h|^2 \varphi^2 .$$

Carrying these estimates in (7.30) we obtain

$$[\lambda_1^{-1} - 2\eta] \iint_{Q_2} |\frac{\partial}{\partial t} [z_m]_h|^2 \varphi^2 dxdt < C(\eta) \iint_{Q_2} |\nabla_x [z_m]_h|^2 [\varphi|\varphi_t| + |\nabla_x \varphi|^2] dxdt +$$

$$+ \frac{1}{\eta} \iint_{Q_2} |[\vec{v}_m \nabla_x K^{-1}(z_m)]_h|^2 {}^2 .$$

Since we have previously shown that $\vec{v}_m$ is locally bounded independent of m the last integral is uniformly bounded with respect to m. Finally we choose $\eta = \lambda_1^{-1}/4$ and recall that $\varphi \equiv 1$ on Q_1. This yields

$$\| \frac{\partial}{\partial t} [z_m]_h \|_{2,Q_1} < C(\| \vec{v}_m \|_{\infty,Q_2}, \mathrm{dist}(Q_1,Q_2), \| \nabla_x u_m \|_{2,\Omega_T}) .$$

Since this estimate is independent of h, from a result of [11] it follows that the weak derivative $\frac{\partial z_m}{\partial t}$ exist and is a locally square summable function in Ω_T. The lemma is proved.

Remark: We stress the following two facts

(a) $\frac{\partial u_m}{\partial t} \in L_2$ only locally, uniformly in m.

(b) The a-priori bound

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{2,\Omega_1} \leq C(\|\vec{v}_m\|_{\infty,\Omega_2}, \mathrm{dist}\,(K,\partial\Omega_T), \|\nabla_x u_m\|_{2,\Omega_T})$$

does not depend upon m, since $\|\vec{v}_m\|_{\infty,\Omega_2}$ and $\|\nabla_x u_m\|_{2,\Omega_T}$ are bounded independently of m, but does depend upon ε via $\|\vec{v}_m\|_{\infty,\Omega_2}$.

7-(b) The limit as $m \to \infty$

We now conclude the proof of Proposition 6.1 by letting $m \to \infty$ in (7.1)-(7.2). From (7.3) it follows that the sequences $\{u_m\}$, $\{\vec{v}_m\}$ are weakly compact in $W_2^{1,0}(\Omega_T)$, and $J_1(\Omega_T)$ respectively; hence subsequences can be selected and relabeled with m, such that

$$u_m \to u_\varepsilon \quad \text{weakly in } W_2^{1,0}(\Omega_T)$$

$$\vec{v}_m \to \vec{v}_\varepsilon \quad \text{weakly in } J_1(\Omega_T) .$$

Lemma 7.11: Let $\varepsilon > 0$ be fixed. There exists a subsequence (relabeled with m) such that

$$u_m \to u_\varepsilon \quad \text{strongly in } L_2(\Omega_T) .$$

Proof of Lemma 7.11.

Let K be a compact of Ω_T. Then by Lemma (7.3) and (7.11) we have

$$\|u_m\|_{\infty,K} + \left\|\frac{\partial u_m}{\partial t}\right\|_{2,K} + \|\nabla_x u_m\|_{2,K} \leq C(K)$$

where C depends upon ε, $\mathrm{dist}(K,\partial\Omega_T)$ but not upon m.

Therefore for a subsequence

$$u_m \to \tilde{u}_\varepsilon \quad \text{strongly in } L_2(K), \text{ and}$$

$$u_m \to \tilde{u}_\varepsilon \quad \text{a.e. in } K .$$

Now by the uniqueness of the weak limit u_ε we have

$$u_\varepsilon \equiv \tilde{u}_\varepsilon, \quad \text{a.e. } (x,t) \in K .$$

Let K_p a sequence of compacts of Ω_T such that $K_p \subset K_{p+1}$ and $\bigcup_{p\geq 1} K_p = \Omega_T$. Then by a diagonalization process a subsequence can be selected and relabeled with m such that

$$u_m \to u_\varepsilon \text{ a.e. in } \Omega_T .$$

Since $\|u_m\|_{\infty,\Omega_T}$ is equibounded, by the Lebesque dominated convergence theorem we have

$$u_m \to u \text{ strongly in } L_2(\Omega_T) .$$

By the trace theorem and (7.3)

$$\text{trace } u_m \to \text{trace } u_\varepsilon \text{ strongly in } L_2(S_T) ,$$

and by monotonicity of $\beta(\cdot)$, $\beta_m(u_m) \to w \in \beta(u)$ weakly in $L_2(\Omega_T)$.

<u>Lemma 7.12:</u> For a subsequence (relabeled with m)

$$\vec{v}_m \to \vec{v}_\varepsilon \text{ strongly in } J(\Omega_T) .$$

<u>Proof of Lemma 7.12.</u> A consequence of Lemma 7.11 is that $\vec{f}(u_m) \to \vec{f}(u_\varepsilon)$ and $H_\varepsilon(u_m) \to H_\varepsilon(u_\varepsilon)$ strongly in $L_2(\Omega_T)$. Therefore since the space dimension is $N = 2$ and $\varepsilon > 0$ is fixed, the strong convergence of $\vec{v}_m$ to $\vec{v}_\varepsilon$ follows by a straightforward adaptation of Serrin's stability theorem (Theorem 6 of [19] page 83).

We can now pass to the limit in (7.1), (7.2) as $m \to \infty$ (for m labeling the particular subsequence chosen above) to obtain the existence of a pair $(u_\varepsilon, \vec{v}_\varepsilon)$ such that

(7.31) $$\|u_\varepsilon\|_{V_2(\Omega_T)}, \quad \|\vec{v}_\varepsilon\|_{J_1^\infty(\Omega_T)} \leq C_0$$

where the constant C_0 does not depend on ε, and u_ε, $\vec{v}_\varepsilon$ satisfy

(7.32) $$\iint_{\Omega_T} \{-\beta(u_\varepsilon)\varphi_t + \nabla_x K(u_\varepsilon)\cdot\nabla_x\varphi + \vec{v}_\varepsilon\cdot\nabla_x u_\varepsilon \varphi\} dxdt = -\int_{S_T} g(x,t,u_\varepsilon)\varphi d\sigma + \int_\Omega \beta(u_0)\varphi(x,0)dx$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $\varphi(x,T) = 0$, and

$$\iint_{\Omega_T} \{-\vec{v}_\varepsilon\cdot\vec{\psi}_t + \nu\nabla_x\vec{v}_\varepsilon : \nabla_x\vec{\psi} + \varepsilon^{-1}H_\varepsilon(u_\varepsilon)\vec{v}_\varepsilon\cdot\vec{\psi}\} dxdt =$$

$$= \iint_{\Omega_T} \vec{f}(u_\varepsilon)\vec{\psi}\, dxdt + \int_\Omega \vec{v}_0(x)\vec{\psi}(x,0)dx$$

for all $\vec{\psi} \in J_{1,1}(\Omega_T)$, $\vec{\psi}(x,T) \equiv 0$.

Statements (b)-(e) are obvious from the estimates we have established in Lemmas 7.3, 7.4, 7.10, 7.2.

8. PROOF OF PROPOSITION 6.2

Since the modulus of continuity has to be uniform in ε we start by listing the estimates that we have which are uniform with respect to ε.

First

(8.1) $$\|u_\varepsilon\|_{\infty,\Omega_T} \le C_2, \qquad \forall\, \varepsilon > 0 .$$

Next

(8.2) $$\|u_\varepsilon\|_{V_2(\Omega_T)},\ \|\vec{v}_\varepsilon\|_{V_2(\Omega_T)} \le C_0 .$$

From the second of (8.2) and Corollary 3.1 we have

(8.3) $$\|\vec{v}_\varepsilon\|_{4,\Omega_T} \le (\text{Const independent of } \varepsilon) .$$

Also the qualitative information in (d) in Proposition 6.1 is essential in order to justify some of the calculations below, but the modulus of continuity will not depend on the local estimates for $\|\frac{\partial}{\partial t} u_\varepsilon\|_{2,K}$.

The function $\omega(\cdot)$ claimed by Proposition 6.2 for $K \subset \Omega_T$ will only depend on the quantities listed in (8.1)-(8.3).

In this section the restriction $N = 2$ will play a dramatic role. The flow of the proof is like the arguments produced in [7, 8]. Now the order of summability (8.3) of $\vec{v}_\varepsilon$ is not high enough to fulfill the assumptions of Theorem 1 of [7]. For this reason a modification in the proof is needed where we will exploit both the dimension $N = 2$ and the particular structure of the equation corresponding to identity (6.1).

Since the arguments have been presented in detail in [7] we will limit ourselves to pointing out the differences that occur at various steps in the proof.

The main idea will consist in showing that given $(x_0,t_0) \in \Omega_T$, we can construct a sequence of cylinders Q_n "centered" at (x_0,t_0), such that $Q_n \supset Q_{n+1}$ and shrinking to

(x_0, t_0), where the oscillation of u_ε decreases, according to the operator associated with (6.1) and in a way determined only by the quantities in (8.1)-(8.3). Given $\eta > 0$, this process will also prescribe the size of a cylinder $Q(\eta)$, where

$$\operatorname*{ess\,osc}_{Q(\eta)} u_\varepsilon < \eta, \qquad \forall\, \varepsilon > 0 .$$

This will obviously yield a modulus of continuity for u_ε over compacts $K \subset \Omega_T$, in an uniform fashion with respect to ε.

[8.A] Preliminary material.

First we report a result of [11] which in our context can be stated as follows. Let u_ε satisfy (6.1) and $u_\varepsilon \in W^{1,1}_{2,loc}(\Omega_T)$. Then u_ε satisfies the integral identity

$$\text{(8.4)} \qquad \int_\Omega \beta(u_\varepsilon)\varphi(x,t)dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \{-\beta(u_\varepsilon)\varphi_t + \nabla_x K(u_\varepsilon)\cdot\nabla_x\varphi + \vec{v}_\varepsilon \nabla_x u_\varepsilon \varphi\}dxdt$$

for all $\varphi \in W_2^{1,1}(\Omega_T)$ such that $x \to \varphi(x,t)$ is supported in Ω for all $t \in [t_1, t_2]$, and all intervals $[t_1, t_2] \subset (0,T]$. We will consider cylinders contained in Ω_T of the following special form. Let (x_0, t_0) be an arbitrary point in Ω_T and denote with $B(R)$ the ball $\{|x - x_0| < R\}$ and with $Q_R(\theta)$ the cylinder

$$Q_R(\theta) \equiv B(R) \times [t_0 - \theta R^2, t_0] .$$

Also if $\sigma_1, \sigma_2 \in (0,1)$ we set

$$Q_R(\theta, \sigma_1, \sigma_2) \equiv B(R - \sigma_1 R) \times [t_0 - \theta(1 - \sigma_2)R^2, t_0] .$$

Consider the definition (4.2) of the graph $\beta(\cdot)$ and set

$$\text{(8.5)} \qquad \beta(u_\varepsilon) = u_\varepsilon + H(u_\varepsilon)$$

where $s \to H(s)$ is the graph

$$\text{(8.6)} \qquad H(s) = \begin{cases} 0 , & s > 0 \\ [-L, 0] , & s = 0 \\ -L , & s < 0 . \end{cases}$$

In (8.4) we employ a test function $\varphi(x,t)$ supported in the ball $B(R)$ for all $t \in [t_0 - \theta R^2, t_0]$ where R and θ are assumed to be so small that $Q_R(\theta) \subset \Omega_T$. By the results of the previous section $\frac{\partial}{\partial t} u_\varepsilon \in L_2(Q_R(\theta))$, therefore substituting (8.5) in (8.4) and integrating by parts with the indicated choice of φ we obtain the identity

$$(8.7)\qquad \int_{B(R)} H(u_\varepsilon)\varphi(x,t)\Big|_{t_0-\theta R^2}^{t} - \int_{t_0-\theta R^2}^{t}\int_{B(R)} H(u_\varepsilon)\varphi_t\,dxdt +$$

$$+ \int_{t_0-\theta R^2}^{t}\int_{B(R)} \{\frac{\partial}{\partial t} u_\varepsilon\varphi + \nabla_x K(u_\varepsilon)\cdot\nabla_x \quad + \vec{v}_\varepsilon\cdot\nabla_x u_\varepsilon\varphi\}dxdt = 0$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(Q_R(\theta))$, and all $t \in [t_0 - \theta R^2, t_0]$. The purpose of (8.7) is to isolate the contribution coming from the jump in $\beta(\cdot)$ with respect to the rest of the equation.

Next we construct particular test functions in (8.7).

Let $(x,t) \to \zeta(x,t)$ be a cutoff function in $Q_R(\theta)$ satisfying

(i) $\zeta(\cdot,t) \in C_0(B(R))$, $|\nabla_x\zeta| \le (\sigma_1 R)^{-1}$

(ii) $\zeta(x,t_0 - \theta R^2) \equiv 0$, $x \in B(R)$, $0 \le \zeta_t \le (\theta\sigma_2 R^2)^{-1}$,

(iii) $\zeta(x,t) \equiv 1$, $(x,t) \in Q_R(\theta,\sigma_1,\sigma_2)$.

Let $k \in \mathbb{R}$ and consider the functions

$$(u_\varepsilon - k)^+ = \max\{u_\varepsilon - k;0\}$$

$$(u_\varepsilon - k)^- = \max\{-(u_\varepsilon - k);0\}\ .$$

It is obvious that if $u_\varepsilon \in L_{r,s}(Q_R(\theta))$, then $(u_\varepsilon - k)^\pm \in L_{r,s}(Q_R(\theta))$, $r,s > 1$. It is known that if $u_\varepsilon \in W_2^{1,1}(Q_R(\theta))$ then also $(u_\varepsilon - k)^\pm \in W_2^{1,1}(Q_R(\theta))$, (cf. [11]).

In (8.7) we will choose

$$\varphi(x,t) = \pm(u_\varepsilon - k)^\pm\zeta^2(x,t)\ .$$

For simplicity of notation we will drop the subscript ε and set

$$-\Phi^\pm(k,t_0) = \int_{B(R)} \pm H(u)(u - k)^\pm\zeta^2(x,t)\Big|_{t_0-\theta R^2}^{t} -$$

$$- \int_{t_0-\theta R^2}^{t}\int_{B(R)} H(u)[\pm(u - k)^\pm\zeta^2(x,\tau)]_t\,dxd\tau\ .$$

The remaining terms in (8.7) are transformed as follows

$$\int_{t_0-\theta R^2}^{t}\int_{B(R)} \pm \frac{\partial}{\partial t} u(u-k)^{\pm}\zeta^2(x,\tau)dxd\tau = \frac{1}{2}\int_{t_0-\theta R^2}^{t}\int_{B(R)} \frac{\partial}{\partial t}[(u-k)^{\pm}]^2\zeta^2(x,\tau)dxd\tau =$$

$$= \frac{1}{2}\int_{B(R)}[(u-k)^{\pm}]^2\zeta^2(x,t)dx - \frac{1}{2}\int_{t_0-\theta R^2}^{t} 2\int_{B(R)}[(u-k)^{\pm}]^2\zeta(x,\tau)\frac{\partial}{\partial t}\zeta\, dxd\tau \geq$$

$$\geq \frac{1}{2}\|(u-k)^{\pm}\zeta\|^2_{2,B(R)}(t) - \|(u-k)^{\pm}(\zeta_t)^{1/2}\|^2_{2,Q_R(\theta)} .$$

For the term involving $\nabla_x\varphi$ we have

$$\int_{t_0-\theta R^2}^{t}\int_{B(R)} \pm\nabla_x K(u)\nabla_x(u-k)^{\pm}\zeta^2 dxd\tau = \int_{t_0-\theta R^2}^{t}\int_{B(R)} K'(u)|\nabla_x(u-k)^{\pm}|^2\zeta^2 +$$

$$+ \int_{t_0-\theta R^2}^{t}\int_{B(R)} 2K'(u)\nabla_x(u-k)^{\pm}\zeta(u-k)^{\pm}\nabla_x\zeta \geq \lambda_0\int_{t_0-\theta R^2}^{t}\int_{B(R)}|\nabla_x(u-k)^{\pm}|^2\zeta^2 dxd\tau -$$

$$- \varepsilon\int_{t_0-\theta R^2}^{t}\int_{B(R)}|\nabla_x(u-k)^{\pm}|^2\zeta^2 dxd\tau - \frac{4\lambda_1^2}{\varepsilon}\iint_{Q_R(\theta)}[(u-k)^{\pm}]^2|\nabla_x\zeta|^2 dxd\tau =$$

$$= (\lambda_0 - \varepsilon)\int_{t_0-\theta R^2}^{t}\int_{B(R)}|\nabla_x(u-k)^{\pm}|^2\zeta^2 dxd\tau - \frac{4\lambda_1^2}{\varepsilon}\|(u-k)^{\pm}|\nabla_x\zeta|\|^2_{2,Q_R(\theta)} .$$

We treat the term involving the velocity as follows

$$\int_{t_0-\theta R^2}^{t}\int_{B(R)} \pm\vec{v}\cdot\nabla_x u[(u-k)^{\pm}]\zeta^2(x,\tau)\,dx\,d\tau =$$

$$= \frac{1}{2}\int_{t_0-\theta R^2}^{t}\int_{B(R)} \vec{v}\cdot(\nabla_x[(u-k)^{\pm}]^2)\zeta^2(x,\tau)\,dx\,d\tau =$$

$$= -\frac{1}{2}\int_{t_0-\theta R^2}^{t} 2\int_{B(R)} \vec{v}[(u-k)^{\pm}]^2\zeta\nabla_x\zeta\,dx\,d\tau .$$

Combining these estimates as parts of (8.7) we obtain

$$\frac{1}{2}\|(u-k)^{\pm}\zeta\|^2_{2,B(R)}(t) + (\lambda_0-\varepsilon)\int_{t_0-\theta R^2}^{t}\int_{B(R)}|\nabla_x(u-k)^{\pm}|^2\zeta^2\,dx\,d\tau \le$$

$$\le \frac{4\lambda_1^2}{\varepsilon}\|(u-k)^{\pm}|\nabla_x\zeta|\|^2_{2,Q_R(\theta)} + \frac{1}{2}\|(u-k)^{\pm}(\zeta_t)^{1/2}\|^2_{2,Q_R(\theta)} +$$

$$+\int_{t_0-\theta R^2}^{t}\int_{B(R)}|\vec{v}||\nabla_x\zeta||(u-k)^{\pm}|^2\,dx\,d\tau + \Phi^{\pm}(k,t_0)$$

for all $t \in [t_0-\theta R^2, t_0]$.

Now choose $\varepsilon = \frac{1}{2}\lambda_0$, recall the construction of the cutoff function $\zeta(x,t)$ and use the arbitrarity of $t \in [t_0-\theta R^2, t_0]$, to conclude that there exists a constant γ depending only upon the data such that

$$\|(u-k)^{\pm}\|^2_{V_2^{1,0}[Q_R(\theta,\sigma_1,\sigma_2)]} \le \gamma[(\sigma_1 R)^{-2} + (\sigma_2\theta R^2)^{-1}]\cdot\|(u-k)^{\pm}\|^2_{2,Q_R(\theta)} +$$

(8.8)

$$+\gamma(\sigma_1 R)^{-1}\iint_{Q_R(\theta)}|\vec{v}|[(u-k)^{\pm}]^2\,dx\,d\tau + \sup_{t\in[t_0-\theta R^2,t_0]}\gamma\Phi^{\pm}(k,t_0)$$

Inequalities (8.8) are valid for every cylinder $Q_R(\theta) \subset \Omega_T$, every pair $\sigma_1, \sigma_2 \in (0,1)$ and every real number k. They are one of the main tools in the proof of continuity. Another tool is the following logarithmic estimate.

[8.B] A logarithmic estimate.

Lemma 8.1: Let $k \in R^+$, $\mu > \operatorname*{ess\,sup}_{Q_R(\theta)} (u - k)^+$ and $0 < \eta < \mu$. Set

$$\psi(x,t) = \ell n^+ \left[\frac{\mu}{\mu - (u-k)^+ + \eta}\right] = \max\left\{\ell n \frac{\mu}{\mu - (u-k)^+ + \eta}, 0\right\}.$$

Then there exists a constant $C = C(\theta)$ such that for all $t \in [t_0 - \theta R^2, t_0]$

$$\int_{B(R-\sigma_1 R)} \psi^2(x,t)dx \le \int_{B(R)} \psi^2(x, t_0 - \theta R^2) + \frac{C}{\sigma_1^2}\left(\ell n \frac{\mu}{\eta}\right) \text{meas } B(R)$$

Remark: For simplicity of notation we will use the same symbol ψ for $\psi(x,t)$ and $\tilde{\psi}(u(x,t))$. In what follows ψ' will mean $\frac{\partial}{\partial u}\tilde{\psi}$.

In the cylinder $Q_R(\theta)$ construct a cutoff function $x \to \zeta(x)$ such that

(i) $\zeta(x) \quad C_0(B(R))$, $|\nabla_x \zeta| < (\sigma_1 R)^{-1}$

(ii) $\zeta(x) \equiv 1$, $x \in B(R - \sigma_1 R)$.

Proof of Lemma 8.1. In (8.7) consider the following test function

$$\varphi(x,t) = (\psi^2)'\zeta^2(x)$$

where $x \to \zeta(x)$ is as above.

It is apparent that $\varphi \in \overset{\circ}{W}{}_2^{1,1}(Q_R(\theta))$ and that $(\psi^2)'' = 2(1 + \psi)(\psi')^2$.

Since $(\psi^2)'$ vanishes at those points $(x,t) \in Q_R(\theta)$ where $(u - k)^+ \le \eta$ and $\eta > 0$, the terms in (8.7) involving $H(u_\varepsilon)$ do not give any contribution. The term involving $\frac{\partial u}{\partial t}$ gives

$$\int_{t_0-\theta R^2}^{t} \int_{B(R)} \frac{\partial}{\partial t} u(\psi^2)'\zeta^2(x)dxd\tau = \int_{B(R)} \psi^2(x,\tau)\zeta^2(x)dx \Big|_{t_0-\theta R^2}^{t}.$$

We estimate the remaining terms as follows

$$\int_{t_0-\theta R^2}^{t} \int_{B(R)} K'(u)\nabla_x u[\{2(1 + \psi)(\psi')^2\}\nabla_x u\zeta^2 + (\psi^2)'\nabla_x \zeta^2]dxdt \geq$$

$$\geq \lambda_0 \int_{t_0-\theta R^2}^{t} \int_{B(R)} 2(1 + \psi)(\psi')^2|\nabla_x u|^2\zeta^2 + J$$

where

$$J = 4\int_{t_0-\theta R^2}^{t} \int_{B(R)} K'(u)\psi\ \psi'\nabla_x u\ \zeta\nabla_x \zeta\ dxd\tau \leq 2\varepsilon \int_{t_0-\theta R^2} \int_{B(R)} (1 + \psi)(\psi')^2|\nabla_x u|^2\zeta^2 +$$

$$+ \frac{8\lambda_1^2}{\varepsilon} \int_{t_0-\theta R^2}^{t} \int_{B(R)} \psi|\nabla_x \zeta|^2 dxd\tau\ .$$

For the term involving the velocities $\vec{v}_\varepsilon$ we have

$$2\int_{t_0-\theta R^2}^{t} \int_{B(R)} \vec{v}\cdot\nabla_x u\psi'\psi\zeta^2 dxd\tau = 2\int_{t_0-\theta R^2} \int_{B(R)} \vec{v}\cdot\nabla_x \psi\cdot\psi\zeta^2 dxd\tau \leq$$

$$\leq 2\varepsilon \int_{t_0-\theta R^2} \int_{B(R)} (1 + \psi)(\psi')^2|\nabla_x u|^2\zeta^2 dxd\tau + \frac{2}{\varepsilon} \int_{t_0-\theta R^2} \int_{B(R)} \psi|\vec{v}|^2\zeta^2 dxd\tau\ .$$

Collecting these estimates we have

$$\|\psi\zeta\|^2_{2,B(R)}(t) + (2\lambda_0 - 4\varepsilon) \int_{t_0-\theta R^2}^{t} \int_{B(R)} (1 + \psi)(\psi')^2|\nabla_x u|^2\zeta^2 dxd\tau \leq$$

(8.9)
$$\leq \|\psi\zeta\|^2_{2,B(R)}(t_0 - \theta R^2) + \frac{8\gamma_1^2}{\varepsilon\gamma_0^2} \int_{t_0-\theta R^2}^{t} \int_{B(R)} \psi|\nabla_x \zeta|^2 dxd\tau +$$

$$+ \frac{2}{\varepsilon} \int_{t_0-\theta R^2}^{t} \int_{B(R)} \psi|\vec{v}|^2\zeta^2 dxd\tau\ .$$

Now choose $\varepsilon = \frac{\lambda_0}{4}$ and observe that $\psi \le \ln \frac{\mu}{\eta}$. Recalling the construction of ζ we conclude that there exists a constant $\tilde{C}$ depending only upon the data such that

$$\int_{B(R-\sigma_1 R)} \psi^2(x,t)dx \le \int_{B(R)} \psi^2(x,t_0 - \theta R^2)dx +$$

(8.10)

$$+ \tilde{C} \ln \frac{\mu}{\eta} \{(\sigma_1 R)^{-2}\theta R^2 \text{ meas } B(R) + \|\vec{v}_\varepsilon\|^2_{4,\Omega_T}[\theta R^2 \text{ meas } B(R)]^{1/2}\} .$$

Since $\|\vec{v}_\varepsilon\|_{4,\Omega_T}$ is uniformly bounded with respect to ε and $[\theta R^2 \text{ meas } B(R)]^{1/2} \le$ $\le$ (const) meas $B(R)$, for $N = 2$, it follows from (8.10) that

(8.11) $$\int_{B(R-\sigma_1 R)} \psi^2(x,t)dx \le \int_{B(R)} \psi^2(x,t_0 - \theta R^2)dx + \frac{C}{\sigma_1^2} (\ln \frac{\mu}{\eta}) \text{ meas } B(R) .$$

The lemma is proved.

Let us return now to the inequalities (8.8) and estimate the term involving velocities as follows.

Set

$$A^{\pm}_{k,R}(\tau) \equiv \{x \in B(R) | (u(x,\tau) - k)^{\pm} > 0\}$$

and

$$M(k,R) = \operatorname*{ess\,sup}_{Q_R(\theta)} (u - k)^{\pm} .$$

Then

$$\gamma(\sigma_1 R)^{-1} \iint_{Q_R(\theta)} |\vec{v}|[(u - k)^{\pm}]^2 dxd\tau \le$$

$$\le \gamma(\sigma_1 R)^{-1}[M(k,R)]^2 \iint_{Q_R(\theta)} |\vec{v}|\chi[(u - k)^{\pm} > 0]dxd\tau$$

where $\chi[(u - k)^{\pm} > 0]$ is the characteristic function of the set $[(u - k)^{\pm} > 0] \cap Q_R(\theta)$.

We have

$$\iint_{Q_R(\theta)} |\vec{v}|\chi[(u - k)^{\pm} > 0]dxd\tau \le \|\vec{v}\|_{4,\Omega_T} \cdot \left[\int_{t_0-\theta R^2}^{t_0} \text{meas } A^{\pm}_{k,R}(\tau)\right]^{3/4}$$

Therefore, by changing the constant γ appropriately, (8.8) can be rewritten as

$$\|(u-k)^{\pm}\|^2_{V_2^{1,0}[Q_R(\theta,\sigma_1,\sigma_2)]} \leq \gamma[(\sigma_1 R)^{-2} + (\sigma_2\theta R^2)^{-1}]\cdot\|(u-k)^{\pm}\|^2_{2,Q_R(\theta)} +$$

(8.12)

$$+\gamma(\sigma_1 R)^{-1}[M(k,R)]^2\left[\int_{t_0-\theta R^2}^{t_0} \text{meas } A^{\pm}_{k,R}(\tau)d\tau\right]^{3/4} + \sup_{t\in[t_0-\theta R^2,t_0]} \gamma\phi^{\pm}(k,t_0) .$$

Let us now show how to conclude the proof of Proposition 6.2.

In [7.8] we demonstrated that the existence of a modulus of continuity for weak solutions of singular quasilinear parabolic equations in divergence form was solely a consequence of inequalities (2.7) page 16 of [7] and lemma 2.2 page 19 of [7]. Now the analog of Lemma 2.2 of [7] is precisely Lemma 8.1 here. Actually the structure of our equation leads to a less complicated logarithmic estimate. We stress the fact that the derivation of Lemma 8.1 employs in an essential way the fact that the dimension N is 2.

As for inequalities (2.7) of [7], their analog here are inequalities (8.12). There is only a slight difference in the term involving $A^{\pm}_{k,R}(\tau)$.

In [7] such term reads

$$I = \left\{\int_{t_0-\theta R^2}^{t_0} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}}d\tau\right\}^{\frac{2}{r}(1+\kappa)}$$

where $q,r > 0$ are linked by

$$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4}$$

and $\kappa \in (0,1)$.

Therefore I can be estimated by

(8.13) $$I \leq \text{Const } R^{N}\cdot R^{N\kappa}$$

In our case the analogous term

$$I' = \gamma\sigma_1^{-1}R^{-1}\left[\int_{t_0-\theta R^2}^{t_0} \text{meas } A^{\pm}_{k,R}(\tau)d\tau\right]^{3/4}[M(k,R)]^2$$

is estimated as

(8.14) $$I' \leq \text{const}\ [M(k,R)]^2 R^N, \qquad (N = 2)\ .$$

Note that in this estimate too $N = 2$ is essential.

Finally let us show the difference between (8.13) and (8.14) does not affect the proof in [7].

Let

$$\mu^+ = \operatorname*{ess\,sup}_{Q_R(\theta)} u; \qquad \mu^- = \operatorname*{ess\,inf}_{Q_R(\theta)} u$$

so that the oscillation ω of u in $Q_R(\theta)$ is

$$\omega = \mu^+ - \mu^-\ .$$

Inequalities (8.12) are employed with the choice of k given by

$$k = \mu^+ - \frac{\omega}{2^s}$$

or

$$k = \mu^- + \frac{\omega}{2^s}$$

where $s \in \mathbb{N}$. Consequently, *from the definition of* $(u - k)^{\pm}$,

$$M(k,R) \leq \frac{\omega}{2^s}\ .$$

Now if $\frac{\omega}{2^s} \leq R^{N\kappa/2}$, the oscillation can be bounded with a power of R, and there is nothing to prove. The case to examine is then when

$$\frac{\omega}{2^s} > R^{N\kappa/2}$$

In [7] we estimated $R^{N\kappa}$ from above with $(\frac{\omega}{2^s})^2$ and carried out the arguments with such an estimate. Now this is precisely the content of (8.14) via (8.15). The term $[M(k,R)]^2$ in (8.14) therefore plays the role of $R^{N\kappa}$ in [7] when division by $(\frac{\omega}{2^s})^2$ is carried out.

We omit the details (which are given in [7]) noticing that our situation is in fact easier due to the simpler structure of the equation. The proof is complete.

<u>Remark:</u> The assumptions $u_0 \in L_\infty(\Omega(0))$ and "$\xi \to g(x,t,\xi)$ monotone at the origin", were used in the proof of lemma 7.3. Now it is apparent from the previous arguments that one only needs

$$u_m \in L_\infty^{loc}(\Omega_T) \text{ independent of } m \text{ and } \varepsilon .$$

The latter can be proved starting from inequalities (8.8) with the aid of Theorem 6.2 of [11] page 105. Consequently one needs only to assume $u_0 \in L_2(\Omega(0))$ and the monotonicity condition on $g(x,t,\cdot)$ at zero can be relaxed.

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This paper considers the time dependent Stefan problem with convection in the fluid phase governed by the Stokes equation, and with adherence of the fluid on the lateral boundaries. The existence of a weak solution is obtained via the introduction of a temperature dependent penalty term in the fluid flow equation, together with the application of various compactness arguments.

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